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PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. XIII.

FROM NOVEMBER, 1881, TO NOVEMBER, 1882.

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PROCEEDINGS

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LONDON MATHEMATICAL SOCIETY.

VOL. XIII.

EIGHTEENTH SESSION, 1881–82.

November 10*th*, 1881.

ANNUAL GENERAL MEETING, held at 22, Albemarle Street.

S. ROBERTS, Esq., F.R.S., President in the Chair.

The Treasurer, (C. W. Merrifield, Esq., F.R.S.), read his Report, which, on the motion of Mr. R. Rawson, seconded by the Rev. R. Harley, was accepted.

At the request of the Chairman, Mr. Harley consented to act as Auditor.

The Secretaries' Report was read and adopted.

From the Report of the Secretaries, it appeared that the number of Members during the past Session had increased from 151 to 159, of whom 53 are compounders; and that the Society had lost by death Mr. T. Cotterill, M.A., formerly a member of the Council, and M. Michel Chasles, the distinguished French geometer.

The communications made to the Society during the past Session had been as follows:

"Considerations respecting the Translation of Series of Observations into Continuous Formulæ:" by Mr. C. W. Merrifield, F.R.S. (Presidential Address.)

"On Bicircular Quartics, with a triple and double Focus and three single Foci, all of them collinear:" by Mr. H. M. Jeffery, F.R.S.

"Note sur la dérivation des déterminants:" by Prof. Teixeira.

"Solution of the Binomial Equation $x^p - 1 = 0$: Quinquisection:" by Prof. Cayley, F.R.S.

"A general Theorem in Kinematics :" by Prof. Minchin.

"On the Solution of the Inverse Logical Problem :" by Mr. W. B. Grove.

"Motion of a Viscous Fluid :" by Mr. T. Craig.

"On the Electrical Capacity of a Conductor bounded by two Spherical Surfaces cutting at any angle :" by Mr. W. D. Niven.

"On an apparently Paradoxical Relation of the Circle, Parabola and Hyperbola :" by Mr. A. J. Ellis, F.R.S.

"A Proof of the Differential Equation which is satisfied by the Hypergeometric Series :" by Rev. T. R. Terry.

"On the Periodicity of Hyperelliptic Integrals of the First Class :" by Mr. W. R. Westropp Roberts.

"On the Tangents drawn from a Point to a Nodal Cubic :" by Mr. R. A. Roberts.

"Sur une propriété du paramètre de la transformée canonique des formes cubiques ternaires :" by Sr. Brioschi.

"Note on a Kinematical Theorem connected with the Rectilinear Courses of two uniformly sailing Vessels :" by Mr. C. W. Merrifield, F.R.S.

"On some Integrals expressible in terms of the first complete Elliptic Integral and of Gamma Functions :" by Mr. J. W. L. Glaisher, F.R.S.

"Some Theorems of Kinematics on a Sphere :" by Mr. E. B. Elliott.

"Supplement on Binomial Biordinals :" by Sir J. Cockle, F.R.S.

"An application of Conjugate Functions :" by Mr. E. J. Routh, F.R.S.

"Note on Abel's Theorem :" by Mr. T. Craig.

"The Equilibrium and Flexure of a Skew Surface :" by Prof. Cayley, F.R.S.

"An application of Elliptic Functions to the Nodal Cubic :" by Mr. R. A. Roberts.

"On the Geodesic Curvature of a Curve on a Surface :" by Prof. Cayley, F.R.S.

"On Operative Symbols in the Differential Calculus :" by Prof. M. W. Crofton, F.R.S.

"Note on the Resolution in Factors of Numbers differing but slightly from π :" by Mr. J. W. L. Glaisher, F.R.S.

"On the Nature of the Quadric represented by the General Equation of the Second Degree in Tetrahedral Coordinates," and "On the five Focal Quadrics of a Cyclide :" by Mr. H. Hart.

"On certain Tetrahedra specially related to Four Spheres meeting in a Point," and "Historical Note on Dr. Graves's Theorem on Confocal Conics :" by the President.

"On Ptolemy's Theorem :" by Mr. C. W. Merrifield, F.R.S.

"The Sumination of certain Hyper-geometric Series:" by Rev. T. R. Terry.

"Quaternion Proof of Mr. S. Roberts's Theorem of 4 co-intersecting Spheres:" by Mr. J. J. Walker.

"Some Solutions of Kirkman's '15 - girl' Problem:" by Mr. E. Carpmael.

"Note on the Coordinates of a Tangent Line to the Curve of Intersection of Two Quadrics:" by Mr. W. R. W. Roberts.

"Sur les Surfaces parallèles:" by Prof. Mannheim.

"On certain Symbolic Operators:" by Mr. J. W. L. Glaisher, F.R.S.

"On a System of Coordinates:" by Prof. Genese.

"Note on a System of Cartesian Ovals passing through Four Points on a Circle:" by Mr. R. A. Roberts.

"On the Gaussian Theory of Surfaces:" by Prof. Cayley, F.R.S.

"On a Theorem in the Calculus of Operations:" by Mr. J. J. Walker.

"On Spherical Quartics, with a Quadruple Cyclic Arc, and a Triple Focus:" by Mr. H. M. Jeffery, F.R.S.

Besides the above, minor communications were made by the President, Messrs. Cayley, Glaisher, Hart, McColl, C. Taylor, H. M. Taylor, and Walker.

The same Journals had been subscribed for as in the preceding Session, and the same exchanges made, with the addition of the Royal Scientific Society of Leipsic.

The Meeting then proceeded to the election of the new Council. The Scrutators (Messrs. Elliott and Leudesdorf) having examined the Balloting Lists, declared the following gentlemen duly elected:—

President, S. Roberts, F.R.S.; Vice-Presidents, Dr. Hirst, F.R.S., J. W. L Glaisher, F.R.S.; Treasurer, C. W. Merrifield, F.R.S.; Hon. Secs., M. Jenkins, M.A., and R. Tucker, M.A.; Other Members—Prof. Cayley, F.R.S., Sir J. Cockle, F.R.S., H. Hart, M.A., Prof. Henrici, F.R.S., A. B. Kempe, F.R.S., Prof. Rowe, M.A., R. F. Scott, M.A., Prof. H. J. S. Smith, F.R.S., H. W. Lloyd Tanner, M.A., and J. J. Walker, M.A.

The Rev. George Pirie, M.A., Professor of Mathematics in the University of Aberdeen, was elected a member of the Society.

The following communications were made:—

"Note on the Limit to the Number of different Proper Fractions whose Denominators are less than x, when x is large:" by Messrs. Jenkins and Merrifield.

"Note on Landen's Theorem:" by Prof. Cayley.

"On the Oscillations of a Viscous Spheroid:" by Prof. Lamb.

"A Geometrical Representation of a System of two Binary Cubics and their Associated Forms:" by Mr. W. R. W. Roberts.

"On the Infinitesimal Bending of Surfaces of Revolution:" by Lord Rayleigh.

"On Tangents to a Cubic forming a Pencil in Involution," by Mr. R. A. Roberts.

The following presents were made:—

Cartes-de-Visite likenesses were received from the Rev. Prof. B. Price, F.R.S., Rev. W. Ellis, Mr. A. Cockshott, Mr. J. R. Harris, Mr. W. H. H. Hudson, Mr. J. Brook Smith, and Dr. J. Casey, F.R.S.

"Physical Society," Proceedings, Vol. iv., Part V., Nov. to Dec., 1881.

"Monatsbericht," Juli, August, September, October, 1881.

"Nautical Almanac" for 1885.

"Bulletin Astronomique et Météorologique de l'Observatoire Impérial de Rio de Janeiro," Juillet 1881, No. 1; Août, No. 2.

"Atti della R. Accademia dei Lincei," Transunti, Vol. vi., Fasc. 1°; Roma, 1881.

"Bulletin des Sciences Math. et Astron.," Tome v., Juillet 1881.

"Transactions of Royal Irish Academy," Vol. xxviii.—Science, vii.—"Extension of the Theory of Screws to the Dynamics of any Material System," by R. S. Ball, LL.D., F.R.S., Dublin, 1881: from the Author.

On the Infinitesimal Bending of Surfaces of Revolution. *By* Lord Rayleigh, F.R.S., Professor of Experimental Physics in the University of Cambridge.

[*Read Nov. 10th*, 1881.]

When a thin sheet of matter is subjected to stress, the force which it opposes to extension is great in comparison with that which it opposes to bending. Under ordinary circumstances, the deformation takes place approximately as if the sheet were inextensible as a whole, a condition which in a remarkable degree facilitates calculation, though (it need scarcely be said) even bending implies an extension of all but the central layers. The inextensibility postulated refers properly to the central layer, which is spoken of as a surface.

We will commence with the case of a surface naturally spherical, and investigate what kind of deformation it admits of, under the condition that no line traced upon the surface is altered in length. The radius of the sphere being a, let the point whose natural coordinates are a, θ, ϕ be displaced to the position $a+\delta r$, $\theta+\delta\theta$, $\phi+\delta\phi$, where δr, $\delta\theta$, $\delta\phi$ are to be treated as small. Since the element of arc ds is of the same length after displacement as before, we have

$$(ds)^2 = a^2(d\theta)^2 + a^2\sin^2\theta\,(d\phi)^2 = (a+\delta r)^2(d\theta+d\delta\theta)^2 + (d\delta r)^2 + (a+\delta r)^2\sin^2(\theta+\delta\theta)(d\phi+d\delta\phi)^2,$$

or, retaining only the first power of δr, $\delta\theta$, $\delta\phi$,

$$a\,d\theta\,d\delta\theta + \delta r\,(d\theta)^2 + \delta r\sin^2\theta\,(d\phi)^2 + a\sin\theta\cos\theta\,\delta\theta\,(d\phi)^2 + a\sin^2\theta\,d\phi\,d\delta\phi = 0.$$

Now

$$\left.\begin{aligned} d\delta\theta &= \frac{d\delta\theta}{d\theta}\,d\theta + \frac{d\delta\theta}{d\phi}\,d\phi \\ d\delta\phi &= \frac{d\delta\phi}{d\theta}\,d\theta + \frac{d\delta\phi}{d\phi}\,d\phi \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

so that

$$(d\theta)^2\left\{\frac{d\delta\theta}{d\theta}+\frac{\delta r}{a}\right\}+d\theta\,d\phi\left\{\frac{d\delta\theta}{d\phi}+\sin^2\theta\frac{d\delta\phi}{d\theta}\right\}$$
$$+(d\phi)^2\left\{\frac{\delta r}{a}\sin^2\theta+\sin\theta\cos\theta\,\delta\theta+\sin^2\theta\frac{d\delta\phi}{d\phi}\right\}=0 \ldots\ldots (2).$$

In this equation $d\theta$ and $d\phi$ are arbitrary, so that the coefficients of $(d\theta)^2$, $d\theta\,d\phi$, and $(d\phi)^2$ must vanish separately. We thus obtain as the conditions of no extension,

$$\frac{d\delta\theta}{d\theta}+\frac{\delta r}{a}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$\frac{d\delta\theta}{d\phi}+\sin^2\theta\frac{d\delta\phi}{d\theta}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

$$\frac{\delta r}{a}+\cot\theta\,\delta\theta+\frac{d\delta\phi}{d\phi}=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

From (3), (4), (5), by elimination of δr,

$$\frac{d}{d\phi}\left(\frac{\delta\theta}{\sin\theta}\right)+\sin\theta\frac{d\delta\phi}{d\theta}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

$$\frac{d}{d\phi}\delta\phi-\sin\theta\frac{d}{d\theta}\left(\frac{\delta\theta}{\sin\theta}\right)\equiv 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7);$$

or, since $\sin\theta\dfrac{d}{d\theta}=\dfrac{d}{d\log\tan\left(\frac{1}{2}\theta\right)}$,

$$\frac{d}{d\phi}\left(\frac{\delta\theta}{\sin\theta}\right)+\frac{d\,\delta\phi}{d\log\tan\left(\frac{1}{2}\theta\right)}=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

$$\frac{d\,\delta\phi}{d\phi}-\frac{d}{d\log\tan\frac{1}{2}\theta}\left(\frac{\delta\theta}{\sin\theta}\right)=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

From (8) and (9) we see that both $\delta\phi$ and $(\sin\theta)^{-1}\,\delta\theta$ satisfy an equation of the second order of the same form, viz.,

$$\frac{d^2u}{d\left(\log\tan\frac{1}{2}\theta\right)^2}+\frac{d^2u}{d\phi^2}=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10).$$

From the nature of the case, u is a periodic function of ϕ, and can be expanded by Fourier's theorem in a series of sines and cosines of ϕ and its multiples. Moreover, each term of the series must satisfy the equations

independently. Thus, if u varies as $\cos s\phi$ or as $\sin s\phi$, (10) becomes

$$\frac{d^2u}{d\,(\log\tan\frac{1}{2}\theta)^2} - s^2u = 0\,;$$

whence
$$u = A'\tan^s\tfrac{1}{2}\theta + B'\cot^s\tfrac{1}{2}\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots(11),$$

where A' and B' are independent of θ.

If we take

$$\delta\phi = \begin{matrix}\cos s\phi\\ \sin s\phi\end{matrix}\left[A\tan^s\tfrac{1}{2}\theta + B\cot^s\tfrac{1}{2}\theta\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots(12),$$

we get for the corresponding value of $\delta\theta$, from (8),

$$\frac{\delta\theta}{\sin\theta} = \begin{matrix}-\sin s\phi\\ +\cos s\phi\end{matrix}\left[A\tan^s\tfrac{1}{2}\theta - B\cot^s\tfrac{1}{2}\theta\right] \quad \ldots\ldots\ldots\ldots(13)\,;$$

and thence, from (5),

$$\frac{\delta r}{a} = \begin{matrix}+\sin s\phi\\ -\cos s\phi\end{matrix}\left[A\,(s+\cos\theta)\tan^s\tfrac{1}{2}\theta + B\,(s-\cos\theta)\cot^s\tfrac{1}{2}\theta\right] \quad \ldots\ldots(14),$$

in which the constants which accompany $\sin s\phi$ are independent of those which accompany $\cos s\phi$.

If we suppose $s = 1$, we get

$$\sin\theta\,\delta\phi = \begin{matrix}+\cos\phi\\ +\sin\phi\end{matrix}\left[A + B - (A-B)\cos\theta\right],$$

$$\delta\theta = \begin{matrix}-\sin\phi\\ +\cos\phi\end{matrix}\left[A - B - (A+B)\cos\theta\right],$$

$$\frac{\delta r}{a} = \begin{matrix}+\sin\phi\\ -\cos\phi\end{matrix}\left[(A+B)\sin\theta\right].$$

The two displacements proportional to $A-B$ are rotations of the whole surface as a rigid body round the axes $\theta = 0$, $\phi = 0$, and $\theta = 0$, $\phi = \frac{1}{2}\pi$. Those proportional to $A+B$ are displacements parallel to the same axes.

The two other motions possible without bending are a rotation round the axis $\theta = 0$, represented by $\delta\theta = 0$, $\delta\phi = \text{const.}$, $\delta r = 0$, and a displacement parallel to the same axis represented by

$$\delta\phi = 0, \quad \frac{d}{d\theta}\left(\frac{\delta\theta}{\sin\theta}\right) = 0, \quad \frac{dr}{a} = -\cot\theta\,\delta\theta,$$

or
$$\delta\phi = 0, \quad \delta\theta = \gamma\sin\theta, \quad \delta r = -\gamma\,a\cos\theta.$$

These correspond to a zero value of s, and are readily obtained from the original equations (3), (4), (5).

If the sphere be complete, the displacements just considered, and corresponding to $s = 0$ and $s = 1$, are the only ones possible. For

higher values of s, we see, from (14), that δr is infinite at one or other pole, unless A and B both vanish. In other words, the complete sphere is perfectly rigid, so far as concerns pure bending.

If neither pole be included in the actual surface, which for example we may suppose bounded by circles of latitude, finite values of both A and B are admissible, and therefore necessary for a complete solution of the problem. But if, as would more often happen, one of the poles, say $\theta = 0$, is included, the constants B must be considered to vanish. Under these circumstances, the solution is

$$\left.\begin{aligned} \delta\phi &= A \tan^s \tfrac{1}{2}\theta \cos s\phi \\ \delta\theta &= -A \sin\theta \tan^s \tfrac{1}{2}\theta \sin s\phi \\ \delta r &= Aa\,(s+\cos\theta) \tan^s \tfrac{1}{2}\theta \sin s\phi \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15),$$

to which is to be added that obtained by writing $s\phi - \frac{1}{2}\pi$ for $s\phi$, and changing the arbitrary constant.

From (15) we see that, along those meridians for which $\sin s\phi = 0$, the displacement is tangential and in longitude only, while along the intermediate meridians for which $\cos s\phi = 0$, there is no displacement in longitude, but one in latitude, and one normal to the surface of the sphere.

Along the equator $\theta = \frac{1}{2}\pi$, and then $\delta\phi = A \cos s\phi$, $\delta\theta = -A \sin s\phi$, $\delta r = Aa\, s \sin s\phi$, so that the maximum displacements in latitude and longitude are equal.

The results embodied in (15) are applicable, even although neither the distribution of thickness nor the form of the boundary are symmetrical with respect to $\theta = 0$, but it is only under this condition that the displacements are *normal* in the mechanical sense. In what follows it will therefore be supposed that the boundary is a circle of latitude, and that the thickness, if variable at all, is a function of latitude only.

In order to find the potential energy of bending, the first step is the purely geometrical one of investigating the curvature at every point of the deformed surface. The principal curvatures differ from the original curvature of the sphere in opposite directions and to an equal amount, and the potential energy of bending corresponding to any element of the surface is proportional to the square of this excess or defect of curvature, without regard to the direction of the principal planes. The calculation of the curvature is somewhat tedious, and it may suffice here merely to give the result. I find that the curvature corresponding to (15) at any point θ, ϕ, and in a plane making with the meridian through the point an angle γ is expressed by

$$\frac{1}{\rho} = \frac{1}{a} - \frac{A}{a} \frac{(s^3 - s) \tan^s \frac{1}{2}\theta}{\sin^2\theta} \sin(s\phi + 2\gamma) \quad \ldots\ldots\ldots (16).$$

The maximum value of $\delta\,(\rho^{-1})$ is therefore the same at every point

along a circle of latitude, and is given by

$$\delta\left(\frac{1}{\rho}\right) = \frac{A\,(s^3-s)\tan^s\frac{1}{2}\theta}{a\sin^2\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots (17),$$

vanishing, as was to be expected, in the cases $s=0$, $s=1$. With respect to the directions of the principal planes, we see, from (16), that along the meridians where $\delta\phi$ vanishes ($\cos s\phi = 0$), the principal planes are the meridian and its perpendicular, while along the meridians where δr vanishes, the principal planes are inclined to the meridian at angles of 45°, as is indeed otherwise obvious.

The potential energy corresponding to the element of surface $a^2\sin\theta\,d\theta\,d\phi$ may be denoted by $a^2 H\left(\delta\,\frac{1}{\rho}\right)^2\sin\theta\,d\theta\,d\phi$, where H depends upon the material and the thickness. The whole potential energy V is to be found by integration with respect to ϕ between the limits 0 and 2π, and for θ between the limits 0 and θ.

We will now prove that the expression for V corresponding to any number of displacements of type (15) involves the squares only, and not the products, of the amplitudes A. In order to have full generality for each value of s, we write, in (15 and 16), $s\phi+\epsilon$ instead of $s\phi$.

We have

$$\frac{1}{\rho}-\frac{1}{a} = -\cos 2\gamma\,\Sigma\,\frac{(s^3-s)\tan^s\frac{1}{2}\theta}{a\sin^2\theta}A\sin(s\phi+\epsilon)$$

$$-\sin 2\gamma\,\Sigma\,\frac{(s^3-s)\tan^s\frac{1}{2}\theta}{a\sin^2\theta}A\cos(s\phi+\epsilon),$$

so that the excess of curvature in the principal plane is given by

$$\left(\delta\frac{1}{\rho}\right)^2 = \left[\Sigma\,\frac{(s^3-s)\tan^s\frac{1}{2}\theta}{a\sin^2\theta}A\sin(s\phi+\epsilon)\right]^2$$

$$+\left[\Sigma\,\frac{(s^3-s)\tan^s\frac{1}{2}}{a\sin^2\theta}A\cos(s\phi+\epsilon)\right]^2.$$

It is now evident that, in the integration with respect to ϕ, the products $A_s A_{s'}$ will disappear in virtue of

$$\int_0^{2\pi}\sin(s\phi+\epsilon)\sin(s'\phi+\epsilon')\,d\phi = 0, \qquad \int_0^{2\pi}\cos(s\phi+\epsilon)\cos(s'\phi+\epsilon)\,d\phi = 0,$$

and the complete expression for V is simply

$$V = 2\pi\,\Sigma\,(s^3-s)^2 A_s^2\int_0^\theta H\sin^{-3}\theta\tan^{2s}\tfrac{1}{2}\theta\,d\theta \quad \ldots\ldots\ldots (18).$$

We will now suppose that H is constant, in order to proceed with the integration. Writing for brevity t for $\tan\frac{1}{2}\theta$, we have

$$dt = \frac{d\theta}{2\cos^2\frac{1}{2}\theta}, \quad \text{so that} \quad d\theta = \frac{2dt}{1+t^2}, \quad \text{and} \quad \sin\theta = \frac{2t}{1+t^2}.$$

Thus $$\int_0^\theta \sin^{-3}\theta \tan^{2s}\tfrac{1}{2}\theta\, d\theta = \tfrac{1}{8}\int (1+t^2)^2\, t^{2s-4}\, dt^2$$

$$= \tfrac{1}{8}\left(\frac{t^{2s-2}}{s-1} + \frac{2t^{2s}}{s} + \frac{t^{2s+2}}{s+1}\right) \quad \text{.....(19).}$$

In the case of a hemisphere $t = 1$, and (19) assumes the form

$$\frac{2s^2-1}{4\,(s^3-s)} \quad \text{.....(20).}$$

Hence, for a hemisphere,

$$V = \tfrac{1}{2}\pi H \Sigma\, (s^3-s)(2s^2-1) A_s^2 \quad \text{.....(21).}$$

If the extreme value of θ be $60°$ instead of $90°$, we get instead of (20)

$$\frac{8s^2+4s-3}{4\,.\,3^{s+1}(s^3-s)} \quad \text{.....(22),}$$

and

$$V = \tfrac{1}{2}\pi H \Sigma\, 3^{-(s+1)}(s^3-s)(8s^2+4s-3) A_s^2 \quad \text{.....(23).}$$

If we wish to take the terms in (15) involving $\cos s\phi$, $\sin s\phi$ separately, we have only to write, in (18), (21), (23), $A_s^2 + A_s'^2$ in place of A_s^2.

These expressions for V, in conjunction with (15), are sufficient for the solution of statical problems, relative to the deformation of spherical shells under the action of given impressed forces. Suppose, for example, that a string of tension F connects the opposite points on the edge of a hemisphere represented by $\theta = \frac{1}{2}\pi$, $\phi = \frac{1}{2}\pi$, $\phi = \frac{3}{2}\pi$, and that it is required to find the deformation. From (15) we see that the work done by the impressed forces corresponding to the deformation δA_s is $\quad -\delta A_s\, as\, (\sin\tfrac{1}{2}s\pi + \sin(\tfrac{1}{2}s\pi + s\pi)\, F.$

This vanishes if s be odd, and if s be even is equal to

$$-2\,\delta A_s\, as \sin\tfrac{1}{2}s\pi\,.\,F.$$

Hence A_s vanishes if s be odd, and, by (21), if s be even,

$$\frac{dV}{dA_s} = \pi H (s^3-s)(2s^2-1)\, A_s = -2as \sin\tfrac{1}{2}s\pi\,.\,F,$$

whence

$$A_s = -\frac{2a\,F \sin\frac{1}{2}s\pi}{\pi H\,(s^2-1)(2s^2-1)} \quad \text{.....(24).}$$

Attaching numerical values to s, we find

$$A_1 = 0, \quad A_2 = \frac{2aF}{\pi H\,.\,3\,.\,7},$$

$$A_3 = 0, \quad A_4 = -\frac{2aF}{\pi H\,.\,15\,.\,31},$$

$$A_5 = 0, \quad A_6 = \frac{2aF}{\pi H\,.\,35\,.\,71} \quad \&c.$$

By (24) and (15), the deformation is completely determined.

If, to take a case in which the force is tangential, we suppose that the hemisphere rests upon its pole with its edge horizontal, and that a rod of weight W is laid symmetrically along the diameter $\theta = \frac{1}{2}\pi$, we find in like manner

$$A_s = \frac{aW \sin \frac{1}{2}s\pi}{\pi H (s^3 - s)(2s^2 - 1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25)$$

for even values of s, and zero for odd values of s. In this case the series is even more convergent than before.

Before we can find the frequencies of free vibration, we must investigate an expression for the kinetic energy. It may readily be proved that only the squares of A_s and A'_s are involved, so that these quantities are really the normal coordinates of the system. We have, from (15),

$$\left(\frac{d\,\delta r}{dt}\right)^2 + \left(\frac{a d\,\delta\theta}{dt}\right)^2 + \left(\frac{a \sin\theta\, d\,\delta\phi}{dt}\right)^2$$

$$= a^2 \left(\frac{dA_s}{dt}\right)^2 \tan^{2s}\tfrac{1}{2}\theta \left\{(s + \cos\theta)^2 \sin^2 s\phi + \sin^2\theta\right\}.$$

When integrated with respect to ϕ, this gives

$$\pi a^2 \left(\frac{dA_s}{dt}\right)^2 \tan^{2s}\tfrac{1}{2}\theta \left\{(s + \cos\theta)^2 + 2\sin^2\theta\right\}.$$

We have now to multiply by $\sin\theta\, d\theta$, and integrate. Let

$$\int_0^\theta \tan^{2s}\tfrac{1}{2}\theta \left\{(s + \cos\theta)^2 + 2\sin^2\theta\right\} \sin\theta\, d\theta = f(s);$$

then, x being written for $1 + \cos\theta$,

$$f(s) = \int^2 \left(\frac{2 - x}{x}\right) \left\{(s-1)^2 + 2x(s+1) - x^2\right\} dx \quad \ldots\ldots\ldots (26).$$

For a hemisphere the lower limit of the integral is unity. If σ denote the surface density, the expression for the whole kinetic energy T is

$$T = \tfrac{1}{2}\pi a^4 \sigma \Sigma f(s) \left\{\left(\frac{dA_s}{dt}\right)^2 + \left(\frac{dA'_s}{dt}\right)^2\right\} \quad \ldots\ldots\ldots\ldots (27).$$

From (27) and (21) the frequencies of free vibration for a hemisphere are immediately obtainable. The equation for A_s, or A'_s, is

$$\sigma a^4 f(s) \frac{d^2 A_s}{dt^2} + H(s^3 - s)(2s^2 - 1) A_s = 0;$$

so that, if A_s varies as $\cos(p_s t + \epsilon)$,

$$p_s^2 = \frac{H(s^3 - s)(2s^2 - 1)}{\sigma a^4 f(s)} \quad \ldots\ldots\ldots\ldots\ldots (28).$$

In like manner for the saucer of 120°, from (22),

$$p_s^2 = \frac{H(s^3-s)(8s^2+4s-3)}{\sigma a^4 f(s) \cdot 3^{s+1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (29).$$

The values of $f(s)$ can be calculated without difficulty in the various cases. Thus, for the hemisphere,

$$f(2) = \int_1^2 x^{-2}(4-4x+x^2)(1+6x-x^2)\,dx$$
$$= 20 \log 2 - 12\tfrac{1}{3} = 1{\cdot}52961,$$
$$f(3) = 57\tfrac{1}{3} - 80 \log 2 = 1{\cdot}88156,$$
$$f(4) = 200 \log 2 - 136\tfrac{1}{3} = 2{\cdot}29609, \text{ \&c.},$$

so that

$$p_2 = \frac{\sqrt{H}}{a^2\sqrt{\sigma}} \times 5{\cdot}2460, \quad p_3 = \frac{\sqrt{H}}{a^2\sqrt{\sigma}} \times 14{\cdot}726, \quad p_4 = \frac{\sqrt{H}}{a^2\sqrt{\sigma}} \times 28{\cdot}462.$$

In experiment, it is the *intervals* between the various tones with which we are most concerned. We find

$$p_3 : p_2 = 2{\cdot}8102, \quad p_4 : p_2 = 5{\cdot}4316.$$

In the case of glass bells, such as are used with air-pumps, the interval between the two gravest tones is usually somewhat smaller; the representative fraction being nearer to 2·5 than 2·8.

For the saucer of 120°, the lower limit of the integral in (26) is $\frac{3}{2}$, and we get on calculation

$$f(2) = {\cdot}12864, \quad f(3) = {\cdot}054884,$$

giving

$$p_2 = \frac{\sqrt{H}}{a^2\sqrt{\sigma}} \times 7{\cdot}9947, \quad p_3 = \frac{\sqrt{H}}{a^2\sqrt{\sigma}} \times 20{\cdot}911,$$
$$p_3 : p_2 = 2{\cdot}6157.$$

The pitch of the two gravest tones is thus decidedly higher than for the hemisphere, and the interval between them is less.

With reference to the theory of tuning bells, it may be worth while to consider the effect of a small change in the angle, for the case of a nearly hemispherical bell. In general

$$p_s^2 = \frac{4H(s^3-s)^2 \int_0^\theta \sin^{-3}\theta \tan^{2s}\tfrac{1}{2}\theta\, d\theta}{a^4\sigma \int_0^\theta \tan^{2s}\tfrac{1}{2}\theta \{(s+\cos\theta)^2 + 2\sin^2\theta\} \sin\theta\, d\theta} \quad \ldots\ldots\ldots\ldots (30).$$

If $\theta = \frac{1}{2}\pi + \delta\theta$, and P_s denote the value of p_s for the exact hemisphere, we get from previous results

$$p_s^2 = P_s^2 \left[1 + \delta\theta \left\{ \frac{4(s^3-s)}{2s^2-1} - \frac{s^2+2}{f(s)} \right\}\right] \quad \ldots\ldots\ldots\ldots (31).$$

Thus $$p_2^2 = P_2^2\left[1+\delta\theta\left\{\frac{24}{7}-\frac{6}{1{\cdot}52961}\right\}\right] = P_2^2\,(1-{\cdot}49\,\delta\theta)$$

$$p_3^2 = P_3^2\left[1+\delta\theta\left\{\frac{96}{17}-\frac{11}{1{\cdot}88156}\right\}\right] = P_3^2\,(1-{\cdot}20\,\delta\theta),$$

shewing that an increase in the angle depresses the pitch. As to the interval between the two gravest tones, we get

$$\left(\frac{p_3}{p_2}\right)^2 = \left(\frac{P_3}{P_2}\right)^2 \times\ (1+{\cdot}29\,\delta\theta),$$

shewing that it increases with θ. This agrees with the results given above for $\theta = 60°$.

The fact that the form of the normal functions is independent of the distribution of density and thickness, provided that they vary only with latitude, allows us to calculate a great variety of cases, the difficulties being merely those of simple integration. If we suppose that only a narrow belt in co-latitude θ has sufficient thickness to contribute sensibly to the potential and kinetic energies, we have simply, instead of (30),

$$p_s^2 = \frac{4H\,(s^3-s)^2\sin^{-4}\theta}{a^4\sigma\{(s+\cos\theta)^2+2\sin^2\theta\}} \qquad \text{.....(32),}$$

whence $$\frac{p_3}{p_2} = 4\sqrt{\left\{\frac{6+4\cos\theta-\cos^2\theta}{11+6\cos\theta-\cos^2\theta}\right\}} \qquad \text{.....(33).}$$

The ratio varies very slowly from 3, when $\theta=0$, to 2·954, when $\theta=\frac{1}{2}\pi$.

If b denote the thickness at any co-latitude θ, $H \propto b^3$, $\sigma \propto b$. I have calculated the ratio of frequencies of the two gravest tones of a hemisphere on the suppositions (1) that $b \propto \cos\theta$, and (2) that $b \propto (1+\cos\theta)$. The formula is that marked (30) with H and σ under the integral signs. In the first case, $p_3 : p_2 = 1{\cdot}7942$, differing greatly from the value for a uniform thickness. On the second more moderate supposition as to the law of thickness, $p_3 : p_2 = 2{\cdot}4591$, $p_4 : p_2 = 4{\cdot}4837$. It would appear that the smallness of the interval between the gravest tones of common glass bells is due in great measure to the thickness diminishing with increasing θ.

It is worthy of notice that the curvature δ (ρ^{-1}), which by (17) varies as $\sin^{-2}\theta\tan^s\frac{1}{2}\theta$, vanishes at the pole for $s = 3$ and higher values, but is finite for $s = 2$.

We will now investigate the deformation of the general surface of revolution under the condition that no line traced upon it suffers extension, and for this purpose we will employ columnar coordinates z, r, ϕ.

We have

$$(ds+d\delta s)^2 = (dz+d\delta z)^2+(r+\delta r)^2\,(d\phi+d\delta\phi)^2+(dr+d\delta r)^2,$$

whence, if $d\delta s$ vanish,

$$dz\, d\delta z + r^2\, d\phi\, d\delta\phi + r\, \delta r\, (d\phi)^2 + dr\, d\delta r = 0.$$

Now
$$d\delta z = \frac{d\delta z}{dz}\, dz + \frac{d\delta z}{d\phi}\, d\phi$$

$$d\delta r = \frac{d\delta r}{dz}\, dz + \frac{d\delta r}{d\phi}\, d\phi,$$

and
$$dr = \frac{dr}{dz}\, dz + \frac{dr}{d\phi}\, d\phi,$$

in which, by hypothesis, $dr/d\phi = 0$.

Thus
$$(dz)^2 \left\{ \frac{d\delta z}{dz} + \frac{dr}{dz}\frac{d\delta r}{dz} \right\} + (d\phi)^2 \left\{ r^2 \frac{d\delta\phi}{d\phi} + r\, \delta r \right\}$$
$$+ dz\, d\phi \left\{ \frac{d\delta z}{d\phi} + r^2 \frac{d\delta\phi}{dz} + \frac{dr}{dz}\frac{d\delta r}{d\phi} \right\} = 0.$$

In this equation the coefficients of $(dz)^2$, $dz\, d\phi$, $(d\phi)^2$ must vanish separately, if the surface is not extended, so that

$$\frac{d\delta z}{dz} + \frac{dr}{dz}\frac{d\delta r}{dz} = 0 \quad \text{.....(34),}$$

$$r \frac{d\delta\phi}{d\phi} + \delta r = 0 \quad \text{.....(35),}$$

$$\frac{d\delta z}{d\phi} + r^2 \frac{d\delta\phi}{dz} + \frac{dr}{dz}\frac{d\delta r}{d\phi} = 0 \quad \text{.....(36).}$$

From these, by elimination of δr,

$$\frac{d\delta z}{dz} - \frac{dr}{dz}\frac{d}{dz}\left(r \frac{d\delta\phi}{d\phi} \right) = 0,$$

$$\frac{d\delta z}{d\phi} + r^2 \frac{d\delta\phi}{dz} - r \frac{dr}{dz}\frac{d^2\delta\phi}{d\phi^2} = 0,$$

from which again, by elimination of δz,

$$\frac{d}{dz}\left(r^2 \frac{d\delta\phi}{dz} \right) - r \frac{d^2 r}{dz^2}\frac{d^2\delta\phi}{d\phi^2} = 0 \quad \text{.....(37).}$$

As in the case of the sphere, if the distribution of thickness and the form of the boundary or boundaries be symmetrical with respect to the axis, the normal functions of the system are found by assuming $\delta\phi$ to be proportional to $\cos s\phi$ or $\sin s\phi$. Geometrically speaking, we are at liberty to make this resolution in any case. Thus the equation for $\delta\phi$ may be put into the form

$$r^2 \frac{d}{dz}\left(r^2 \frac{d\delta\phi}{dz} \right) + s^2 r^3 \frac{d^2 r}{dz^2}\, \delta\phi = 0 \quad \text{.....(38).}$$

The simplest application of these results is to the *cylinder* for which r is constant, equal say to a. Thus (34), (35), (36), (38) become simply

$$\frac{d\delta z}{dz} = 0, \quad \delta r + a\frac{d\delta\phi}{d\phi} = 0, \quad \frac{d\delta z}{d\phi} + a^2\frac{d\delta\phi}{dz} = 0 \quad \ldots\ldots(39),$$

$$\frac{d^2\delta\phi}{dz^2} = 0 \quad \ldots\ldots(40).$$

By (40), if $\delta\phi \propto \cos s\phi$, we may take

$$a\delta\phi = (A_s a + B_s z)\cos s\phi \quad \ldots\ldots(41),$$

and then, by (39),

$$\delta r = s(A_s a + B_s z)\sin s\phi \quad \ldots\ldots(42),$$

$$\delta z = -s^{-1} B_s a \sin s\phi \quad \ldots\ldots(43).$$

Corresponding terms, with fresh arbitrary constants, obtained by writing $s\phi + \frac{1}{2}\pi$ for $s\phi$, may of course be added. If $B_s = 0$, the displacement is in two dimensions only. This kind of deformation is considered more at length in my book on the "Theory of Sound," § 233.

If an inextensible disc be attached to the cylinder at $z = 0$, so as to form a kind of cup, the displacements δr and $\delta\phi$ must vanish for that value of z, exception being made of the case $s = 1$. Hence $A_s = 0$, and

$$a\delta\phi = B_s z\cos s\phi, \quad \delta r = s B_s z \sin s\phi, \quad \delta z = -s^{-1} B_s a \sin s\phi \ldots\ldots(44).$$

In the case of a *cone*, for which $r = \tan\gamma \,.\, z$, the equations become

$$\left.\begin{aligned} \frac{d\delta z}{dz} + \tan\gamma\frac{d\delta r}{dz} = 0, \quad z\tan\gamma\frac{d\delta\phi}{d\phi} + \delta r = 0 \\ \frac{d\delta z}{d\phi} + z^2\tan^2\gamma\frac{d\delta\phi}{d\phi} + \tan\gamma\frac{d\delta r}{d\phi} = 0 \end{aligned}\right\} \quad \ldots\ldots(45),$$

$$\frac{d}{dz}\left(z^2\frac{d\delta\phi}{dz}\right) = 0 \quad \ldots\ldots(46).$$

If we take, as usual, $\delta\phi \propto \cos s\phi$, we get as the solution of (46)

$$\delta\phi = (A_s + B_s z^{-1})\cos s\phi \quad \ldots\ldots(47),$$

and corresponding thereto

$$dr = s\tan\gamma\,(A_s z + B_s)\sin s\phi \quad \ldots\ldots(48),$$

$$\delta z = \tan^2\gamma\,[s^{-1} B_s - s(A_s z + B_s)]\sin s\phi \quad \ldots\ldots(49).$$

If the cone be complete up to the vertex at $z = 0$, $B_s = 0$, so that

$$d\phi = A_s\cos s\phi \quad \ldots\ldots(50),$$

$$dr = sA_s r\sin s\phi \quad \ldots\ldots(51),$$

$$dz = -sA_s\tan\gamma\, r\sin s\phi \quad \ldots\ldots(52).$$

For the cone and the cylinder, the second term in the general equation (38) vanishes. We shall obtain a more extensive class of soluble cases

by supposing that the surface is such that

$$r^3 \frac{d^2r}{dz^2} = \text{constant} \quad \ldots\ldots(53),$$

an equation which is satisfied by surfaces of the second degree in general. If

$$\frac{z^2}{a^2} + \frac{r^2}{b^2} = 1 \quad \ldots\ldots(54),$$

we shall find

$$r^3 \frac{d^2r}{dz^2} = -\frac{b^4}{a^2} \quad \ldots\ldots(55),$$

and thus (38) takes the form

$$\frac{d^2\delta\phi}{da^2} - \frac{s^2b^4}{a^2}\delta\phi = 0 \quad \ldots\ldots(56),$$

if $\delta\phi \propto \cos s\phi$, and α is defined by

$$\alpha = \int r^{-2}\, dz \quad \ldots\ldots(57),$$

or in the present case

$$\alpha = \frac{a}{2b^2} \log \frac{a+z}{a-z} \quad \ldots\ldots(58).$$

The solution of (56) is

$$\delta\phi = \left[A \left(\frac{a+z}{a-z}\right)^{-\frac{1}{2}s} + B \left(\frac{a+z}{a-z}\right)^{+\frac{1}{2}s} \right] \cos s\phi \quad \ldots\ldots(59).$$

The corresponding values of δr and dz are to be obtained from (35) and (36).

If the surface be complete through the vertex $z = a$, the term multiplied by B must disappear. Thus, omitting the constant multiplier, we may take

$$\delta\phi = \left(\frac{a-z}{a+z}\right)^{\frac{1}{2}s} \cos s\phi \quad \ldots\ldots(60);$$

whence, by (35), (36),

$$\delta r = n \frac{b}{a} \frac{(a-z)^{\frac{1}{2}s+\frac{1}{2}}}{(a+z)^{\frac{1}{2}s-\frac{1}{2}}} \sin s\phi \quad \ldots\ldots(61),$$

$$\delta z = (sz+a) \frac{b^2}{a^2} \frac{(a-z)^{\frac{1}{2}s}}{(a+z)^{\frac{1}{2}s}} \sin s\phi \quad \ldots\ldots(62).$$

If we measure z' from the vertex, $z' = a - z$, and

$$\delta\phi = \left(\frac{z'}{r}\right)^s \cos s\phi \quad \ldots\ldots(63),$$

$$\delta r = sr \left(\frac{z'}{r}\right)^s \sin s\phi \quad \ldots\ldots(64),$$

$$\delta z = -\delta z' = \frac{b^2}{a^2} \left\{ (s+1)\, a - sz' \right\} \left(\frac{z'}{r}\right)^s \sin s\phi \quad \ldots\ldots(65),$$

For the parabola, a and b are infinite, while $b^2/a = 2a'$, and $r^2 = 4a'x'$.

Thus
$$\left.\begin{aligned} \delta\phi &= r^s \cos s\phi \\ \delta r &= s r^{s-1} \sin s\phi \\ \delta x' &= -2\,(s+1)\,a' r^s \sin s\phi \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(66).$$

A Geometrical Representation of a System of two Binary Cubics and their Associated Forms.

By W. R. Westropp Roberts, M.A.

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1. The object of this paper is to invest with a certain geometrical meaning the algebraic forms arising in a system of two binary cubics; that is, to construct geometrically points which shall represent the linear, quadratic, cubic, and quartic covariants of the system, and to express the vanishing of invariants by geometrical relations connecting such points. We may consider any binary quantic as derived from a system of three surfaces by assuming

$$X = \phi_1 (x_1 x_2), \quad Y = \phi_2 (x_1 x_2), \quad Z = \phi_3 (x_1 x_2), \quad W = \phi_4 (x_1 x_2),$$

equations which in themselves imply, by elimination of x_1 and x_2, two fixed relations between X, Y, Z, W, denoting a fixed curve in space, while the given binary quantic equated to zero, enables us to obtain a third such relation. The transformation here employed is one in which ϕ_1, ϕ_2, ϕ_3, ϕ_4, are cubic functions of x_1 and x_2. By linear transformation, this substitution is reducible to

$$X = x_1^3, \quad Y = x_1^2 x_2, \quad Z = x_1 x_2^2, \quad W = x_2^3,$$

the fixed curve in this case being evidently a twisted cubic. The equation of an osculating plane of the curve, the parameter of the corresponding point of which is $x_1 : x_2$, being (Salmon's *Geometry of Three Dimensions*, Art. 368)

$$X x_2^3 - 3Y x_1^2 x_2^2 + 3Z x_1^2 x_2 - x_1^3 W = 0,$$

the parameters answering to osculating planes through any point O, the coordinates of which are X', Y', Z', W', are given by the equation

$$x_2^3 X' - 3x_2^2 x_1 Y' + 3x_1 x_2^2 Z' - x_1^3 W' = 0,$$

the points of contact lying in the plane

$$XW' - 3YZ' + 3ZY' - WX' = 0.$$

But this plane passes through O, the given point. To any plane there corresponds a point O, the point of intersection of the osculating planes

at the points where it meets the curve, a point which plays an important part in the following investigation. It follows readily, from the last equation, that if a point O lie in the plane corresponding to a point O', then O' lies in the plane corresponding to O, and the line joining O and O' possesses a certain invariant relation to the curve. Also, the locus of corresponding points of planes passing through a given line is a right line, which may be called the corresponding line to the given one, their relation being reciprocal.

(2.) Let us consider the binary cubic

$$f = a_0 x_1^3 + 3a_1 x_1^2 x_2 + 3a_2 x_1 x_2^2 + a_3 x_2^3 = a_x^3 = b_x^3 = \&c.,$$

adopting Clebsch's notation. By our transformation, the binary cubic is transformed into a plane F, the equation of which is

$$a_0 X + 3a_1 Y + 3a_2 Z + a_3 W = 0.$$

Now it can be easily shown that the corresponding point of this plane is given by the equations

$$X = a_3, \quad Y = -a_2, \quad Z = a_1, \quad W = -a_0.$$

We shall call it the point O.

(3.) It is known that through any point O in space can be drawn one chord meeting the curve in two points. Let us now determine these points, being given the point O.

The coordinates of the line joining the points on the curve, the parameters of which are $x_1 : x_2$, $y_1 : y_2$, respectively, are easily found to be

$$\begin{aligned} a &= R_0 R_2, & f &= 4R_1^2 - R_0 R_2, \\ b &= 2R_1 R_2, & g &= -2R_1 R_0, \\ c &= R_2^2, & h &= R_0^2, \end{aligned}$$

where $$R_0 = x_2 y_2, \quad -2R_1 = x_1 y_2 + y_1 x_2, \quad R_2 = x_1 y_1.$$

Now take two equations of the chord, viz.,

$$aX + bY + cZ = 0,$$

$$hY - gZ + aW = 0,$$

and these furnish us with

$$R_0 X + 2R_1 Y + R_2 Z = 0,$$

$$R_0 Y + 2R_1 Z + R_2 W = 0.$$

If we now suppose $X = a_3$, $Y = -a_2$, $Z = a_1$, $W = -a_0$, the equation determining the parameters of the two points, in which the chord through O meets the curve, is

$$(a_0 a_2 - a_1^2) x_1^2 + (a_0 a_3 - a_1 a_2) x_1 x_2 + (a_1 a_3 - a_2^2) x_2^2 = 0.$$

But this is the equation of the Hessian of f, or

$$\Delta_x^2 = \Delta_0 x_1^2 + 2\Delta_1 x_1 x_2 + \Delta_2 x_2^2 = 0.$$

Thus, then, to the Hessian of f correspond the two points in which the chord through O meets the curve.

(4.) I shall now show that the plane F passes through the line of intersection of the osculating planes at the two Hessian points on the curve. To prove this, let us find the equation of the plane through O and this line.

Let X', Y', Z', W'; X'', Y'', Z'', W'' be the coordinates of the two Hessian points respectively, then the equation of such a plane must be of the form

$$\lambda (XW' - 3YZ' + 3ZY' - WX') - \mu (XW'' - 3YZ'' + 3ZY'' - WX'') = 0,$$

where $$\lambda = a_0 X'' + 3a_1 Y'' + 3a_2 Z'' + a_3 W'',$$

$$\mu = a_0 X' + 3a_1 Y' + 3a_2 Z' + a_3 W'.$$

Remembering that $$x_1 y_1 = a_1 a_3 - a_2^2,$$

$$-(x_1 y_2 + y_1 x_2) = a_0 a_3 - a_1 a_2,$$

$$x_2 y_2 = a_0 a_2 - a_1^2,$$

we find, dividing by a factor $(x_1 y_2 - y_1 x_2)$, that the equation becomes

$$R (a_0 X + 3a_1 Y + 3a_2 Z + a_3 W) = 0,$$

R being the discriminant of the binary cubic.

Hence the plane F passes through the line of intersection of the osculating planes at the Hessian points. If, then, the chord through O meet the curve in real points, the plane F must meet it in two imaginary points, since the binary cubic is then the difference of two cubes, and has but one real factor.

(5.) Let us now investigate the plane

$$\lambda (XW' - 3YZ' + 3ZY' - WX') + \mu (XW'' - 3YZ'' + 3ZY'' - WX'') = 0,$$

and we find, after a few obvious reductions, that it becomes

$$R \{(a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3) X + 3 (a_0 a_1 a_3 + a_1^2 a_2 - 2a_0 a_2^2) Y$$
$$+ 3 (2a_1^2 a_3 - a_0 a_2 a_3 - a_1 a_2^2) Z + (3a_1 a_2 a_3 - a_0 a_3^2 - 2a_2^3) W\} = 0;$$

but this is the cubic covariant plane Q.

We shall denote its corresponding point by ω.

We see, then, that the planes F and Q are harmonically conjugate with regard to the osculating planes at the Hessian points, and that, since the Hessian of the binary Q_x^3 has the same factors as Δ_x^2, the corresponding point ω of the plane Q lies on the chord through O, and is the harmonic conjugate of O with regard to the Hessian points.

Hence to the cubic covariant Q_x^3 corresponds the plane through the intersection of the osculating planes at the Hessian points and the harmonic conjugate on the chord through O of the same point with regard to the Hessian points. Again, if the point O lie at one side of the developable generated by tangent lines to the curve from which a real chord can be drawn, two of the roots of the binary cubic are imaginary; if the point O lies on the developable, two roots are equal; and if at the other side from which a real chord cannot be drawn, all the roots are real.

(6.) We can now discuss the system of two binary cubics and their associated forms, and shall adopt the notation of Clebsch in our investigation. Let, then, f and ϕ denote the two cubics, Δ and ∇ their Hessians, Q and K their cubic covariants, R and P their discriminants.*

For convenience, I give a list of the invariants and covariants of a system of two cubics, of which there are, according to Clebsch and Gordan, 28 forms in all, and which I discuss geometrically.

Prof. Cayley has, however, drawn my attention to the fact that two of the linear covariants (xa^4a^3 and xa^3a^4) of Clebsch and Gordan, have been shown to be non-fundamental by Prof. Sylvester. See Sylvester's "Tables of the Generating Functions," *American Journal of Mathematics*, t. ii. (1879).

Table of Covariants of a System of Two Cubics.

7 Invariants—

$$aa\ ;\quad a^4\ ,\quad a^3a\ ,\quad a^2a^2\ ,\quad aa^3\ ,\quad a^4\ :\quad a^3a^3\ .$$
$$(aa)^3;\ (\Delta\Delta')^2,\ (\Delta\Theta)^2,\ (\Delta\nabla)^2,\ (\nabla\Theta)^2,\ (\nabla\nabla')^2 :\ (\Theta\Delta)(\Theta\nabla)(\Delta\nabla).$$

6 Linear Covariants—

$$xa^2a\ ,\quad xaa^2\ ;\quad xa^4a\ ,\quad xa^3a^2\ ,\quad xa^2a^3\ ,\quad xaa^4\ .$$
$$(\Delta a)^2a_x,\ (\nabla a)^2a_x;\ (a\Delta)^2(a\Delta')\Delta'_x,\ (Q\nabla)^2Q_x,\ (K\Delta)^2K_x,\ (a\nabla)^2(a\nabla')\nabla'_x.$$

6 Quadratic Covariants—

$$x^2a^2,\ x^2aa,\ x^2a^2,\quad x^2a^3a\ ,\quad x^2a^2a^2\ ,\quad x^2aa^3\ .$$
$$\Delta_x^2,\ \Theta_x^2,\ \nabla_x^2,\ (\Delta\Theta)\Delta_x\Theta_x,\ (\Delta\nabla)\Delta_x\nabla_x,\ (\nabla\Theta)\nabla_x\Theta_x.$$

6 Cubic Covariants—

$$x^3a,\ x^3a;\ x^3a^3,\quad x^3a^2a\ ,\quad x^3aa^2\ ,\ x^3a^3.$$
$$a_x^3,\ \ a_x^3\ ;\ \ Q_x^3,\ \ (\Delta a)\Delta_x a_x^2,\ \ (\nabla a)\nabla_x a_x^2,\ \ K_x^3\ .$$

1 Quartic Covariant—

$$x^4aa\ .$$
$$(aa)a_x^2a_x^2\ .$$

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* The reader is referred to Clebsch's "Theorie der Binären Algebraischen Formen, § 61.—Vollständiges System zweier Cubischen Formen."

In this table I have identified, at the suggestion of Prof. Cayley, the covariants given by Sylvester with the notation of Clebsch and Gordan.

We have then

$$f \equiv a_0x_1^3 + 3a_1x_1^2x_2 + 3a_2x_1x_2^2 + a_3x_2^3 = a_x^3 = b_x^3 = \&c.,$$

$$\phi = a_0x_1^3 + 3a_1x_1^2x_2 + 3a_2x_1x_2^2 + a_3x_2^3 = \alpha_x^3 = \beta_x^3 = \&c.$$

There is one quartic form $(a\alpha)\,a_x^2\alpha_x^2$ which we shall first discuss. The coordinates of the lines of intersection of the planes F and Φ are

$$a = a_0\alpha_3 - a_3\alpha_0, \qquad f = 9\,(a_1\alpha_2 - a_2\alpha_1),$$
$$b = 3\,(a_1\alpha_3 - a_3\alpha_1), \quad g = 3\,(a_2\alpha_0 - a_0\alpha_2),$$
$$c = 3\,(a_2\alpha_3 - a_3\alpha_2), \quad h = 3\,(a_0\alpha_1 - a_1\alpha_0)\ ;$$

while the coordinates of the corresponding line, or the line joining the the points O and O', are

$$a = a_1\alpha_2 - a_2\alpha_1, \qquad f = a_0\alpha_3 - a_3\alpha_0,$$
$$b = a_1\alpha_3 - a_3\alpha_1, \qquad g = a_2\alpha_0 - a_0\alpha_2,$$
$$c = a_2\alpha_3 - a_3\alpha_2, \qquad h = a_0\alpha_1 - a_1\alpha_0.$$

Now the coordinates of a tangent line to the curve are

$$a = x_1^2x_2^2, \qquad f = 3x_1^2x_2^2,$$
$$b = -2x_1^3x_2, \qquad g = 2x_1x_2^3,$$
$$c = x_1^4, \qquad h = x_2^4.$$

Forming, then, the well-known condition that the line (F, Φ) may meet a tangent line, we find

$$x_1^4\,(a_0\alpha_1 - a_0\alpha_1) + 2x_1^3x_2\,(a_0\alpha_2 - a_2\alpha_0) + x_1^2x_2^2\,\{a_0\alpha_3 - a_3\alpha_0 + 3\,(a_1\alpha_2 - a_2\alpha_1)\}$$
$$+ 2x_1x_2^3\,(a_1\alpha_3 - a_1\alpha_3) + x_2^4\,(a_2\alpha_3 - a_3\alpha_2) = 0,$$

or
$$(a\alpha)\,a_x^2\alpha_x^2 = 0.$$

Hence to the Jacobian of f and ϕ correspond the four points on the curve, the tangents at which meet the line F, Φ. From the values above given of the coordinates of the line OO', we see that the four tangents also meet this line. The Jacobian $(a\alpha)\,a_x^2\alpha_x^2$ is thus geometrically shown to be a combinantive covariant, since it depends only on the position of the line (F, Φ).

(7.) In addition to the forms f and ϕ, and their covariants Q and K, there are two forms of the third degree in the variables

$$(a\nabla)\,a_x^2\nabla x, \qquad (a\Delta)\,a_x^2\Delta_x,$$

which we now discuss. Since the cubic covariant is the evectant of the discriminant, it follows easily that the cubic covariant plane Q is

the polar plane of O with regard to the developable. By taking the polar plane of a point on the line OO', we find it to be of the form

$$\lambda^3 Q + \lambda^2 \mu q + \alpha \mu^2 k + \mu^3 K = 0.$$

Hence we have two new planes q and k, giving rise to two cubic covariants in the binary system, the leading terms of which are

$$q_0 = a_0 \frac{dQ_0}{da_0} + a_1 \frac{dQ_0}{da_1} + a_2 \frac{dQ_0}{da_2} + a_3 \frac{dQ_0}{da_3},$$

$$k_0 = a_0 \frac{dK_0}{da_0} + a_1 \frac{dK_0}{da_1} + a_2 \frac{dK_0}{da_2} + a_3 \frac{dK_0}{da_3},$$

where

$$Q_0 = a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3,$$

$$K_0 = a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3.$$

Now it is easy to show that

$$q_0 = 3\{a_0 (a_0 a_3 - a_1 a_2) - 2a_1 (a_0 a_2 - a_1^2)\}$$
$$+ a_0 \{a_0 a_3 - a_0 a_3 - 3 (a_1 a_2 - a_1 a_2)\}$$
$$= 6 (a_0 \Delta_1 - a_1 \Delta_0) + (a\alpha)^3 a_0 ;$$

hence

$$q_x = 6 (a\Delta)\, a^2 \Delta_x + (a\alpha)^3 f,$$

and in the same way $k_x = 6 (a\nabla)\, a_x \nabla_x - (a\alpha)^3 \phi$.

We have now expressed q_x^3 and k_x^3 in terms of Clebsch's forms, and can represent them as follows. The covariant q_x^3 is transformed into a plane which is the polar plane of O' with regard to the polar quadric of O.

In like manner, k_x^3 is transformed into a plane which is the polar plane of O with regard to the polar quadric of O'. These theorems are immediate algebraic consequences of the method of generation of q and k.

(8.) We now discuss Clebsch's two linear covariants p_x and π_x. If through a point X, Y, Z, W, and two points on the curve the parameters of which are determined by the equation

$$R_0 x_1^2 + 2R_1 x_1 x_2 + R_2 x_2^2 = R_x^2 = 0,$$

we draw a plane, it will meet the curve in a third point determined by the equation

$$x_1 (R_0 Y + 2R_1 Z + R_2 W) + x_2 (R_0 X + 2R_1 Y + R_2 Z) = 0,$$

and if X, Y, Z, W be the corresponding point of a plane resulting by transformation from a binary cubic form ρ_x^3, the above equation reduces to $(R\rho)^2 \rho_x = 0$. Now let us suppose $R_x^2 = \Delta_x^2$, and $\rho_x^3 = \phi$, we find

$$(\Delta a)^2 a_x = 0, \quad \text{or} \quad p_x = 0.$$

Hence the linear covariant p_x is represented by the point in which a plane through O', and containing the chord through O, meets the curve, a similar construction representing π_x; where $\pi_x = (\nabla a)^2 a_x$.

(9.) We now turn to the quadratic covariants.

The equation determining the parameters of the points in which a chord through any point on the line OO' meets the curve, is

$$\lambda^2 \Delta + 2\lambda\mu\Theta + \mu^2 \nabla = 0.$$

The forms Δ and ∇ have been discussed before, and it remains to attach a geometrical meaning to $\Theta_x^2 \equiv (a\alpha)^2 a_x \alpha_x$. Now, from any given point on the curve can be drawn two chords to meet the line OO', the above equation in $\lambda : \mu$ determining the points of meeting on OO'; if, then, the coefficient of $\lambda\mu$ vanishes, the chords drawn from the given point meet the line OO' in two points harmonically conjugate with regard to O and O'.

Hence Θ is represented geometrically by the two points on the curve, the chords from which to meet the line OO' divide it harmonically.

(10.) Let us now determine the tangent lines to the curve which meet a line through X, Y, Z, W, and a point on the curve the parameter of which is $x_1' : x_2'$.

The coordinates of a line through X, Y, Z, W, and a point $x_1' : x_2'$ on the curve, are

$$\begin{aligned} a &= x_1'^2 x_2'^2 Z - x_1' x_2'^2 Y, & f &= x_1'^3 W - X x_2'^3, \\ b &= x_1' x_2'^2 X - x_1'^3 Z, & g &= x_1'^2 x_2' W - X x_2'^3, \\ c &= x_1'^3 Y - x_1'^2 x_3' X, & h &= x_1 x_2'^2 W - Z x_2'^3. \end{aligned}$$

Forming the condition that this line may meet a tangent line, the coordinates of which are given by (6), and dividing by a factor $(x_1 x_2' - x_2 x_1')^2$, we find

$$x_1^2 (W x_1' - Z x_2') + 2 x_1 x_2 (Y x_2' - Z x_1') + (Y x_1' - X x_2') x_2^2 = 0.$$

Suppose now the point (X, Y, Z, W) to be the point O, and the point on the curve the point p_x or $(\Delta a)^2 a_x$, the above equation becomes

$$x_1^2 (a_0 x_1' + a_1 x_2') + 2 x_1 x_2 (a_1 x_1' + a_2 x_2') + x_2^2 (a_2 x_1' + a_3 x_2') = 0,$$

or

$$(ap)\, a_x^2 = 0.$$

We can easily express this in terms of Clebsch's forms as follows:

$$\Theta_x^2 = (a\alpha)^2 a_x \alpha_x,$$

hence

$$2 (\Theta\Delta)\, \Theta_x = (a\alpha)^2 \{a_x (a\Delta) + \alpha_x (a\Delta)\},$$

or

$$\begin{aligned} 2 (\Theta\Delta)\, \Theta_x \Delta_x &= (a\alpha)\, (a\Delta)(a\alpha)\, \Delta_x \alpha_x + (a\alpha)^2 (a\Delta)\, a_x \Delta_x \\ &= a_x\, (a\alpha)(a\Delta)\{a_x (\Delta\alpha) + \alpha_x (a\Delta)\} + (a\alpha)^2 (a\Delta)\, a_x \Delta_x \\ &= -a_x\, (a\alpha)(\Delta\alpha)^2 \text{ (See Clebsch, § 34).} \end{aligned}$$

Hence $$(pa)\, a_x^2 = 2\,(\Theta\Delta)\,\Theta_x\,\Delta_x.$$

The three remaining quadratic covariants are constructed as follows: Through O and the point p on the curve draw a line; then the two points on the curve, the tangents at which meet this line, represent the quadratic covariant $(\Theta\Delta)\,\Theta_x\,\Delta_x$. In like manner, the covariant $(\Theta\nabla)\,\Theta_x\,\nabla_x$ is represented, while the form $(\Delta\nabla)\,\Delta_x\nabla_x$ is represented by the points the tangents at which meet the line $O\pi$ or the line $O'p$.

(11.) We shall now discuss the six remaining linear covariants. We saw that the linear covariant obtained by drawing a plane through the corresponding point of a plane resulting by transformation from a binary form ρ_x^3, and two points on the curve given by the equation $R_x^2 = 0$, was $(R\rho)^2\,\rho_x$. If we substitute K for ρ and Δ for R, we obtain $(\Delta K)^2\,K_x$, which is Clebsch's form.

To represent the covariant $(\Delta K)^2\,K_x$, draw a plane through ω', the corresponding point of the plane K, and containing the chord through O: this plane will meet the curve in the required point. By a similar construction the covariant $(\nabla Q)^2\,Q_x$ is represented.

(12.) We now show how to represent the forms

$$(\pi\nabla)\,\nabla_x, \quad \text{and} \quad (p\Delta)\,\Delta_x.$$

We have shown how to represent the two points corresponding to the form $(\Theta\Delta)\,\Theta_x\,\Delta_x$, which we shall call for the moment R_x^2, and if in the form $(R\rho)^2\,\rho_x$ we put a for ρ, we have $(Ra)^2\,a_x$.

Now $$R_x^2 = (\Theta\Delta)\,\Theta_x\,\Delta_x,$$

therefore $$(Ra)^2 = (\Theta\Delta)(\Theta a)(\Delta a);$$

but $$(a\Delta)\,a_y\,a_x\,\Delta_y = (a\Delta)a_y^2\,\Delta_x,$$

hence $$(Ra)^2\,a_x = (a\Delta)(a\Theta)^2\,\Delta_x.$$

Now $$p_x = -2\,(\Theta a)^2\,a_x,$$

therefore $$(p\Delta)\,\Delta_x = -2\,(\Theta a)^2\,(a\Delta)\,\Delta_x,$$

or $$(Ra)^2\,a_x = -\tfrac{1}{2}\,(p\Delta)\,\Delta_x.$$

Hence draw a plane through O and the two points given by the equation $(\Delta\Theta)\,\Delta_x\,\Theta_x = 0$, and already constructed, which will meet the curve in a third point representing the linear covariant $(p\Delta)\,\Delta_x$. In the same way the form $(\pi\nabla)\nabla_x$ is represented.

The two remaining linear covariants may be represented as follows. Suppose a plane drawn through O' and containing the chord through O, this plane will meet the curve in points given by the equation $\Delta_x^2\,p_x$, which we shall call $\rho_x^3 = \Delta_x^2\,p_x$; through the corresponding point of this plane, and the chord through O', draw a plane which will

meet the curve in a third point given by the equation

$$(\rho\nabla)^2\rho_x = 0.$$

Now $$\rho_y^3 = \Delta_y^2\, p_y,$$

therefore $$3\rho_y^2\rho_x = 2\Delta_y\Delta_x\, p_y + \Delta_y^2 p_x,$$

or $$3\,(\rho\nabla)^2\rho_x = 2\,(\Delta\nabla)\,\Delta_x\,(p\nabla) + (\Delta\nabla)^2 p_x,$$

which expresses $(\rho\nabla)^2\rho_x$ in terms of Clebsch's forms.

In the same way we can represent the form

$$2\,(\pi\Delta)(\Delta\nabla)\,\nabla_x + (\nabla\Delta)^2\pi_x.$$

(13.) We now turn to the invariants of which there are seven, two being combinantive. We shall first discuss the combinants. Let us seek the condition that the point O may lie in the plane Φ, and we know, by (1), that when this condition is fulfilled O' lies in the plane F.

Expressing the condition that the point $(a_3, -a_2, a_1, -a_0)$ may lie in the plane $$a_0X + 3a_1Y + 3a_2Z + a_3W = 0,$$

we find $$(aa)^3 = 0 \quad \text{or} \quad J = 0.$$

Hence J vanishes when O lies in the plane Φ; but in this case the line (F, Φ) becomes identical with its corresponding line OO', and this relation is evidently combinantive, since it depends only on the position of the line (F, Φ).

Clebsch denotes by $\Omega = 0$ the condition that must be satisfied in order that $f + k\phi$ may become a perfect cube; hence, when it vanishes, it will be possible to draw an osculating plane through the line (F, Φ). In this case, it is easy to see that the corresponding line meets the curve; and, for the same reason as before, this relation is combinantive.

Now Clebsch has shown analytically that $2\Omega = (p\pi)$, and we can show geometrically that when Ω vanishes, or the line OO' meets the curve in a point O'', that the points p and π coincide.

The point p is the point in which a plane containing O' and the chord through O meets the curve again; but this plane contains the line OO', which by hypothesis meets the curve in O''; hence, when $\Omega = 0$, p coincides with O''; and in like manner π is shown to coincide with O'', and therefore with p.

The two discriminants have been discussed before as the conditions that the points O and O' should lie on the developable, respectively.

We now come to the invariants

$$\Sigma = (aK)^3 = (\Theta\nabla)^2, \quad S = (aQ)^3 = (\Theta\Delta)^2.$$

Now it is easy to see that Σ is the condition that O should lie on the plane K, or that the line (F, K) should coincide with its corresponding line. S, in like manner, is the condition that O' should lie in the plane Q.

It remains to discuss the invariant $T = (\Delta\nabla)^2$. The vanishing of this invariant may be expressed in different ways; when $T = 0$, the planes through any chord and the four points in which the chords through O and O' meet the curve, form a harmonic system; again, when $T = 0$, the polar quadric of O passes through O', and *vice versâ*.

In this investigation I have adopted throughout the notation and procedure of Clebsch, as it lends itself more readily than other methods to the identification of binary forms with their geometrical significations.

NOTE.—Since this paper was written, I have found an article in the *Quarterly Mathematical Journal*, by Prof. Cayley, Vol. x, "On a Geometrical Interpretation of a Binary Cubic and its Covariants," in which Prof. Cayley gives to the Hessian of the cubic a geometrical interpretation which is practically the same as that given in this paper.

On Tangents to a Cubic forming a Pencil in Involution.

By R. A. ROBERTS, B.A.

[*Read Nov. 10th*, 1881.]

I propose to determine, in this paper, the locus of the vertex of a pencil in involution circumscribed about a non-singular cubic curve. The condition that a binary sextic should have its roots in involutions being of the fifteenth degree in the coefficients, it follows that the locus is of the forty-fifth degree (see Salmon's "Higher Plane Curves," Art. 94, and "Higher Algebra," Art. 251). The method which I adopt to obtain the equation of the locus will show that it breaks up into twelve cubics and nine lines.

I commence by showing that, if a binary sextic be written in the form $u^2 - kv^3$, where u is a cubic, and v a quadratic, and M the skew invariant of u and v vanish (Salmon's "Higher Algebra," Art. 198), then the roots of the sextic are in involution. If x, y be the factors of the Hessian of u, we may write $u \equiv x^3 + y^3$, $v \equiv ax^2 + by^2 + 2hxy$, when the condition $M = 0$ gives $a^3 - b^3 = 0$. Taking $a = b$, and putting $x^2 + y^2 = u_1$, $xy = u_2$, the sextic can be written

$$(u_1 + 2u_2)\,(u_1 - u_2)^2 - k\,(au_1 + 2hu_2)^3,$$

showing that the roots are in involution.

Referring the cubic to a canonical triangle, we have

$$U \equiv x^3 + y^3 + z^3 + 6mxyz,$$

and the tangential equation of the curve is then

$$\Sigma \equiv \alpha^6+\beta^6+\gamma^6-(2+32m^3)\,(\beta^3\gamma^3+\gamma^3\alpha^3+\alpha^3\beta^3) \\ -24m^2\alpha\beta\gamma\,(\alpha^3+\beta^3+\gamma^3)-(24m+48m^4)\,\alpha^2\beta^2\gamma^2 = 0,$$

which may be written in the form

$$\{\alpha^3+\beta^3-(1+16m^3)\,\gamma^3-12m^3\alpha\beta\gamma\}^2-4\,(1+8m^3)\,(\alpha\beta+2m\gamma^2)^3 = 0,$$

and, of course, also in the similar forms

$$\{\beta^3+\gamma^3-(1+16m^3)\,\alpha^3-12m^3\alpha\beta\gamma\}^2-4\,(1+8m^3)\,(\beta\gamma+2m\alpha^2)^3 = 0,$$

$$\{\gamma^3+\alpha^3-(1+16m^3)\,\beta^3-12m^3\alpha\beta\gamma)^2-4\,(1+8m^3)\,(\gamma\alpha+2m\beta^2)^3 = 0;$$

and, since there are three other canonical triangles, we see that Σ can be written in the form u^2-kv^3 in nine other ways, or altogether in twelve ways.

It is evident that one of these curves of the third class and the corresponding conic have six inflexional tangents of the cubic for common tangents.

Let us apply the condition $M=0$ to the form $\Sigma \equiv u^2-kv^3$, where

$$u \equiv \alpha^3+\beta^3-(1+16m^3)\,\gamma^3-12m^3\alpha\beta\gamma, \quad v \equiv \alpha\beta+2m\gamma^2,$$

and

$$\alpha x+\beta y+\gamma z = 0.$$

Now, when M vanishes, the linear covariant obtained by substituting differential symbols in v, and operating on u, is a factor of u. Substituting, then, $y\frac{d}{d\gamma}-z\frac{d}{d\beta}$, $z\frac{d}{d\alpha}-x\frac{d}{d\gamma}$, $x\frac{d}{d\beta}-y\frac{d}{d\alpha}$ for α, β, γ respectively in v, and operating on u, we obtain

$$\alpha\,(2my^2-2m^2zx)+\beta\,(2mx^2-2m^2yz)+\gamma\,\{2m^2z^2+(1+24m^3)\,xy\} = 0,$$

which combined with $\alpha x+\beta y+\gamma z = 0$ gives

$$\alpha : \beta : \gamma = \; 4m^2yz^2+(1+24m^3)\,xy^2-2mx^2z \\ : 2my^2z-(1+24m^3)\,x^2y-4m^2xz^2 : 2m\,(x^3-y^3).$$

From these values we deduce

$$\alpha+\beta+4m^2\gamma = \quad (y-x)\,\{4m^2z^2+(1+16m^3)\,xy + \quad (2myz-8m^3x^2) \\ + \quad (2mzx-8m^3y^2)\},$$

$$\vartheta\alpha+\vartheta^2\beta+4m^2\gamma = (\vartheta y-\vartheta^2x)\,\{4m^2z^2+(1+16m^3)\,xy+\vartheta\,(2myz-8m^3x^2) \\ +\vartheta^2(2mzx-8m^3y^2)\},$$

$$\vartheta^2\alpha+\vartheta\beta+4m^2\gamma = (\vartheta^2y-\vartheta x)\{4m^2z^2+(1+16m^3)\,xy+\vartheta^2(2myz-8m^3x^2) \\ +\vartheta\,(2mzx-8m^3y^2)\},$$

where ϑ is a cube root of unity. But u may be written in the form

$$(\alpha+\beta+4m^2\gamma)\,(\vartheta\alpha+\vartheta^2\beta+4m^2\gamma)\,(\vartheta^2\alpha+\vartheta\beta+4m^2\gamma)-(1+8m^3)^2\,\gamma^3 = 0;$$

hence, substituting for $\alpha+\beta+4m^2\gamma$, &c., their values in terms of x, y, z, and dividing by x^3-y^3 (which represents three harmonic polars),

we obtain

$$\{4m^2z^2+(1+16m^3)\,xy\}^3+(2myz-8m^3x^2)^3+(2mzx-8m^3y^2)^3$$
$$-3\{4m^2z^2+(1+16m^3)\,xy\}\,(2myz-8m^3x^2)\,(2mzx-8m^3y^2)$$
$$+8m^3\,(1+8m^3)^2\,(x^3-y^3)^2=0,$$

which, being multiplied out and arranged according to powers of z, becomes

$$64m^6z^6+768m^7xyz^4+(192m^6+8m^3)\,(x^3+y^3)\,z^3$$
$$+2{,}304m^8x^2y^2z^2+(1{,}152m^7+48m^4)\,(x^3+y^3)\,xyz$$
$$+(128m^6+8m^3)\,(x^3+y^3)^2+(1+8m^3)^2\,x^3y^3=0,$$

or
$$64m^6\,(x^3+y^3+z^3+6mxyz)^2+8m^3\,(1+8m^3)\,(x^3+y^3)$$
$$\times(x^3+y^3+z^3+6mxyz)+(1+8m^3)^2\,x^3y^3=0;$$

and this equation breaks up into the factors

$$(1+8m^3)\,x^3+8m^3U=0,\quad(1+8m^3)\,y^3+8m^3U=0,$$

putting $x^3+y^3+z^3+6mxyz=U$. We have, of course, by symmetry, the cubic $(1+8m^3)\,z^3+8m^3U=0$. If the equation of U referred to another canonical triangle be written $X^3+Y^3+Z^3+6m'XYZ=0$, where $X=x+y+z$, $Y=x+\vartheta y+\vartheta^2z$, $Z=x+\vartheta^2y+\vartheta z$, we have $m'=\dfrac{1-m}{2m+1}$, and the equation of one of the loci is

$$(1+8m'^3)\,X^3+8m'^3\,(X^3+Y^3+Z^3+6m'XYZ)=0,$$

or
$$(1+8m^3)\,(x+y+z)^3-8\,(m-1)^3\,U=0,$$

and the two other cubics are

$$(1+8m^3)\,(x+\vartheta y+\vartheta^2z)^3-8\,(m-1)^3\,U=0,$$
$$(1+8m^3)\,(x+\vartheta^2y+\vartheta z)^3-8\,(m-1)^3\,U=0.$$

Similarly, by means of the other canonical triangles, we find the equations

$$(1+8m^3)\,(y+z+\vartheta x)^3-8\,(m-\vartheta)^3\,U=0,$$
$$(1+8m^3)\,(z+x+\vartheta y)^3-8\,(m-\vartheta)^3\,U=0,$$
$$(1+8m^3)\,(x+y+\vartheta z)^3-8\,(m-\vartheta)^3\,U=0;$$
$$(1+8m^3)\,(y+z+\vartheta^2x)^3-8\,(m-\vartheta^2)^3\,U=0,$$
$$(1+8m^3)\,(z+x+\vartheta^2y)^3-8\,(m-\vartheta^2)^3\,U=0,$$
$$(1+8m^3)\,(x+y+\vartheta^2z)^3-8\,(m-\vartheta^2)^3\,U=0.$$

Thus we have twelve cubics in all, which together with the nine harmonic polars make up the locus of the forty-fifth degree.

December 8th, 1881.

S. ROBERTS, Esq., F.R.S., President, in the Chair.

Mr. George Henry Stuart, M.A., late Fellow of Emmanuel College, Cambridge, was elected a Member, and Miss C. A. Scott was admitted into the Society.

The Auditor (Rev. R. Harley) read his Report. On the motion of the Chairman, a vote of thanks was unanimously accorded to the Auditor for his services.

The following papers were read:—

"On the Polar Planes of Four Quadrics:" by Mr. W. Spottiswoode, Pres. R.S.

"On some Forms of Cubic Determinants:" by Mr. R. F. Scott.

"On the Flow of a Viscous Fluid through a Pipe or Channel:" by Prof. Greenhill.

"The Covariant which is the complete Locus of the Vertex of the Involution-pencil of Tangents to a Cubic:" by Mr. J. J. Walker.

The following presents were received:—

"American Journal of Mathematics," Vol. iv., No. 1.

Cartes-de-Visite likenesses from Rev. A. J. C. Allen, and Mr. Donald McAlister.

On the Polar Planes of Four Quadrics. By William Spottiswoode, M.A., P.R.S.

[*Read Dec. 8th*, 1881.]

Consider the four quadric surfaces U', U_1', U_2', U_3', where $U' = (x', y', z', t')^2$, &c.; and let P', or (x, y, z, t), be any point in space. Then the equation of the tangent plane through the point P to the surface U' may be written thus,

$$u'x + v'y + w'z + s't = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where $$(\delta_x, \delta_y, \delta_z, \delta_t)\ U' = u', v', w', s',$$

and similarly for the tangent planes through P to the other surfaces. Again, if $U', \ldots u, \ldots$ be the same functions of x, y, z, t, that $U', \ldots u', \ldots$ are of x', y', z', t'; then the equations of the polar plane of the point P with respect to the quadric U' may be written thus:

$$ux' + vy' + wz' + st' = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

with similar expressions for the polar planes of the point P with respect to the three other quadrics.

The condition that these planes may meet in a point P' will be found by eliminating x', y', z', t' from the four equations of the form (2). It will, in fact, be represented by the Jacobian of the four surfaces, viz.,

by the equation
$$\begin{vmatrix} u, & u_1, & u_2, & u_3 \\ v, & v_1, & v_2, & v_3 \\ w, & w_1, & w_2, & w_3 \\ s, & s_1, & s_2, & s_3 \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3);$$

and, as the constituents of this determinant are linear in the coordinates of the point P, it follows that, in order that the four polar planes meet in a point, the point P of which they are polars must lie on the quartic surface of which the equation is (3). This surface, and a kindred surface called the symmetroid, have been discussed by Cayley in his first memoir on "Quartic Surfaces," in the *Proceedings of the London Mathematical Society*, Vol. III., p. 19.

The coordinates of the point of intersection P' may be found from any three of the equations represented by (2). Thus, if the developed

form of (3) be $\quad Au + Bv + Cw + Ds = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4),$

we shall have
$$\left.\begin{matrix} x' : y' : z' : t' \\ = A : B : C : D \end{matrix}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

in which we may substitute for A, B, C, D [which are the minors formed from the last three columns of the matrix (3)] the minors formed from any three columns of the same matrix.

If the point P' be restricted to a given plane Π, say the plane whose

equation is $\quad lx' + my' + nz' + pt' = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (6),$

then, on substituting for x', y', z', t', from (3), we shall have

$$lA + mB + nC + pD = 0,$$

or, writing down, for the sake of brevity, only the upper line of the

determinant, $\quad | \, l, u_1, u_2, u_3 \, | = 0 \ldots\ldots\ldots\ldots\ldots\ldots (7).$

That is to say, the point P must lie on the curve of intersection of the cubic surface (7) with the quartic surface (3). But, since all the determinants that can be formed from the matrix

$$l, \; u, \; u_1, \; u_2, \; u_3$$

will vanish, if any two of them vanish, it follows that the two equations (3) and (7) may be replaced by any other two, *e.g.*, by the following

$$| \, l, u, u_1, u_2 \, | = 0, \quad | \, l, u, u_1, u_3 \, | = 0 \quad \ldots\ldots\ldots\ldots (8).$$

In other words, the point P must lie on the curve of intersection of the

two cubic surfaces (8). The order of this curve is 6. See Salmon's "Higher Algebra," 3rd Edition, pp. 243 *et seqq.* In fact, the five surfaces represented by the disjunctive equation

$$\| l,\ u,\ u_1,\ u_2,\ u_3 \| = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

have a common curve of intersection.

It may be further observed that the equations (3) and (7) are both satisfied, if the equations

$$\begin{vmatrix} u_1, & v_1, & w_1, & s_1 \\ u_2, & v_2, & w_2, & s_2 \\ u_3, & v_3, & w_3, & s_3 \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

are satisfied. In other words, the surfaces (3) and (7) have in common the curve whose equations are (10). But, in the same way as before, the coordinates of the point P' will be proportional to minors formed from any three of the columns of the matrix (9); and, as the equations (10) are equivalent to

$$A = 0,\ B = 0,\ C = 0,\ D = 0,$$

it follows that, for every position of the point P on the curve, the coordinates of the point P' become indeterminate. Similarly, the surfaces (8) have in common the curve whose equations are

$$\begin{vmatrix} l, & m, & n, & p \\ u, & v, & w, & s \\ u_1, & v_1, & w_1, & s_1 \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11),$$

and, for every position of the point P on this curve, the coordinates of the point P' become indeterminate.

Again, if the point P' be restricted to a given straight line, say the intersection of the plane Π with a plane Π_1, whose equation is

$$l_1 x' + m_1 y' + n_1 z' + p_1 t' = 0,$$

then, in addition to the condition (7), we shall have the condition

$$| l_1,\ u_1,\ u_2,\ u_3 | = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12).$$

But the three equations (3), (7), (12) may be replaced by any three independent equations derived from the matrix

$$l,\ l_1,\ u,\ u_1,\ u_2,\ u_3,$$

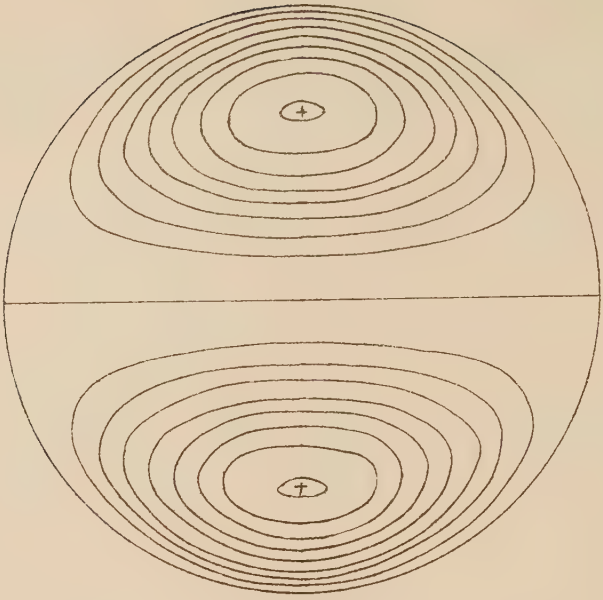
e.g., by the equations

$$| l, l_1, u, u_1 | = 0, \quad | l, l_1, u, u_2 | = 0, \quad | l, l_1, u, u_3 | = 0 \ldots (13);$$

and consequently the point P must lie at the intersection of the three quadrics (13). In fact, the fifteen surfaces represented by the disjunctive equation

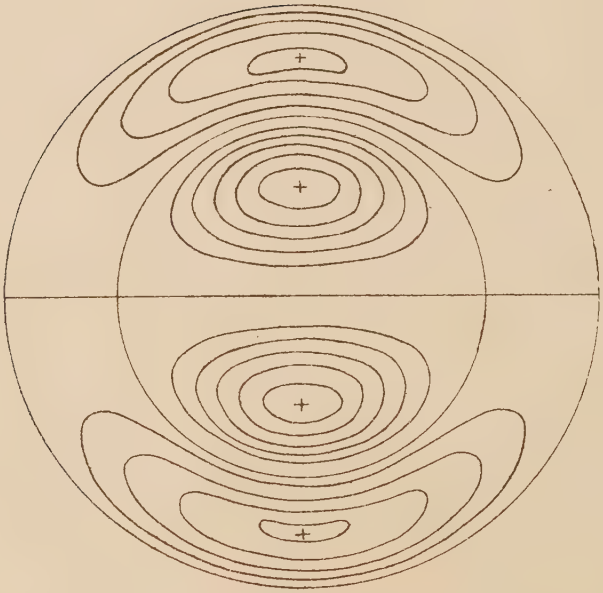
$$\| l, l_1, u, u_1, u_2, u_3 \| = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

FIG. 1.



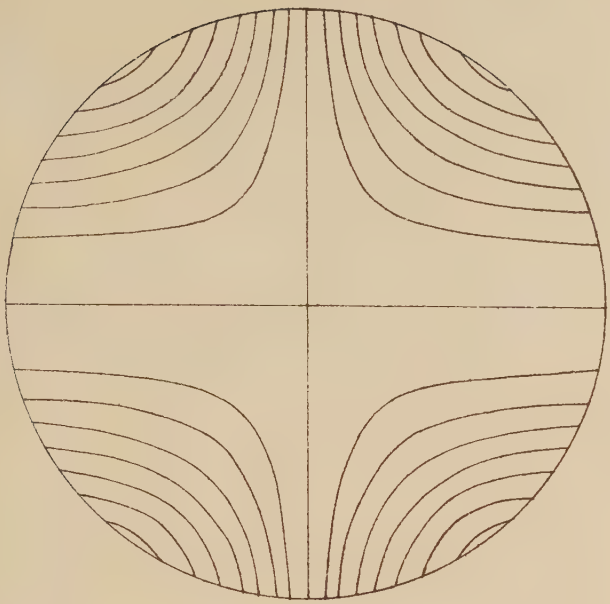
[$n=1$; $s=1$.]

FIG. 2.



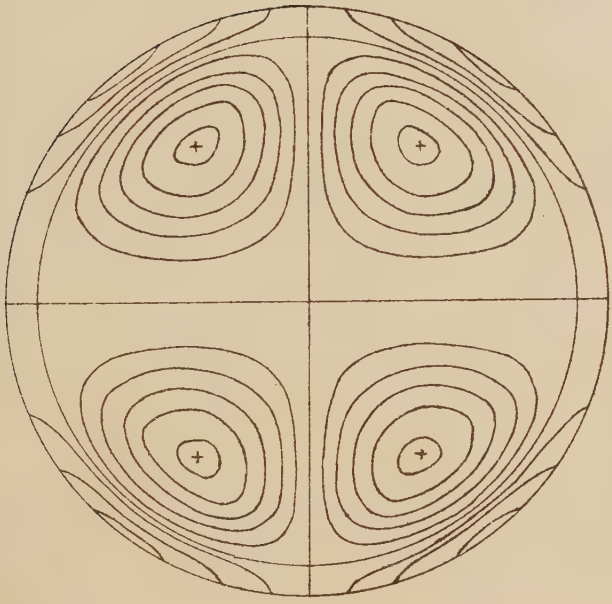
[$n=2$; $s=2$.]

FIG. 3.



[$n=2$; $s=1$.]

FIG. 4.



[$n=1$; $s=2$.]

intersect in the same common points. The number of these points is 4. See Salmon's "Higher Algebra," *l. c.*

Again, the three equations (13) will be satisfied if the equations

$$\begin{vmatrix} l, & m, & n, & p \\ l_1, & m_1, & n_1, & p_1 \\ u, & v, & w, & s \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

are satisfied. In other words, the quadrics have in common the straight line whose equations are (15). If a, b, c, f, g, h, be six coordinates of the line of intersection of the planes Π, Π_1, then the line may be defined by any two of the equations

$$\left.\begin{aligned} L &= \quad . \quad hv - gw + as = 0 \\ M &= -hu \quad . \quad + fw + bs = 0 \\ N &= \quad gu - fv \quad . \quad + cs = 0 \\ P &= \quad au + bv + cw \quad . \quad = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (16).$$

But, since the coordinates of the point P' are given by the equations

$$\left.\begin{aligned} x' : y' : z' : t' \\ = L : M : N : P \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (17),$$

it follows that, for every position of the point P on the line (16), the coordinates of the point P' become indeterminate. In fact, if P lies anywhere on the line $\Pi\,\Pi_1$, then P may lie anywhere on the line (17).

With the quartic surface (3), and with the other surfaces which have similarly occurred in this paper, there are connected certain cubic curves in space; for example, the curve represented by the equations

$$\begin{vmatrix} u_1, & u_2, & u_3 \\ s_1, & s_2, & s_3 \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18).$$

These, as is well known, represent a cubic curve in space, because the two equations, say $u_1 s_2 - u_2 s_1 = 0$, $u_1 s_3 - u_3 s_1 = 0$, are the equations of two quadric surfaces having a common generator $u_1 = 0$, $s_1 = 0$. We may enquire, in what points does this curve meet the surface (3). If we put (18) into the following form,

$$u_1 : s_1 = u_2 : s_2 = u_3 : s_3 = U : S \text{ suppose} \quad \ldots\ldots\ldots\ldots (19),$$

it will be found that the quartic breaks up into two factors, viz., the expression (3) becomes

$$(uS - sU)\,(v_2 w_3 - v_3 w_2 + v_3 w_1 - v_1 w_3 + v_1 w_2 - v_2 w_1) = 0 \quad \ldots\ldots (20);$$

and, if we take the first as the vanishing factor and combine it with (18), we obtain

$$\begin{vmatrix} u, & u_1, & u_2, & u_3 \\ s, & s_1, & s_2, & s_3 \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21);$$

and, if further we put these equations into the following form

$$u = \lambda s, \quad u_1 = \lambda s_1, \quad u_2 = \lambda s_2, \quad u_3 = \lambda s_3,$$

and eliminate x, y, z, t, we shall obtain a biquadratic in λ. The four roots of this equation, successively substituted in (21), will give four points in which the cubic curve meets the surface (3). Eight more points are to be found by combining (18) with the second factor of (20); and the remainder are to be accounted for by the fact that the quadrics (18) have a common generator.

When the point P' is restricted to the plane Π, the cubic curve in question can meet the curve represented by (3) and (7) only under a condition. The cubic (7) breaks up into the factors

$$(lS-pU)(v_2w_3-v_3w_2+\ldots)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(22).$$

Taking $uS-sU$, $lS-pU$, as the vanishing factors, we obtain

$$u:s=u_1:s_1=u_2:s_2=u_3:s_3=l:p \quad \ldots\ldots\ldots\ldots\ldots\ldots(23),$$

and if from these we eliminate x, y, z, t, we obtain the required condition. If, however, we suppose the plane Π to be no longer entirely arbitrary, but to be determined so that the constant $l:p$ shall satisfy the condition in question, it will follow that there are four series of positions of the plane for which the cubic curve (18) will meet the curve of intersection of the surfaces (3) and (7). Beside the constant $l:p$, there are two others $m:p$, $n:p$, which may also be determined so as to satisfy required conditions. We may, in fact, determine them so that the two other cubic curves

$$\begin{vmatrix} v_1, & v_2, & v_3 \\ s_1, & s_2, & s_3 \end{vmatrix} = 0, \quad \begin{vmatrix} w_1, & w_2, & w_3 \\ s_1, & s_2, & s_3 \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots(24)$$

may meet the curve represented by the equations (3), (7). The plane will be then completely determined.

When the point P' is restricted to the line (Π, Π_1), there will be two conditions to be fulfilled in order that the cubic curve may pass through the points defined by the equations (14). In fact, we shall, on substituting as before, again find the equations (21) and (22); and, in addition to them, also the following:

$$(l_1S-p_1U)(w_2s_3-w_3s_2+\ldots)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(25).$$

But the equation for determining $l_1:p_1$ will be the same as that for determining $l:p$. If, therefore, we take for $l:p$, and $l_1:p_1$, two different roots of that equation, we may satisfy both conditions.

On some Forms of Cubic Determinants. By Mr. R. F. Scott.

[*Read December 8th*, 1881.]

1. If we form the strata of a cubic determinant C from the square arrays

$$\begin{matrix} a_{11} \dots a_{1n} \\ \dots \quad \dots \\ a_{n1} \dots a_{nn} \end{matrix} \qquad \begin{matrix} b_{11} \dots b_{1n} \\ \dots \quad \dots \\ b_{n1} \dots b_{nn} \end{matrix}$$

by repeating the first array m times, and the second $n-m$ times, then [it being remembered that the general cubic determinant of order n $|\ a_{ijk}\ |$ can be represented by the product of n lineo-linear functions of alternate units, such as

$$\begin{aligned} & a_{i11}\, e_1 \epsilon_1 + a_{i12}\, e_1 \epsilon_2 + \dots + a_{i1n}\, e_1 \epsilon_n \\ + & a_{i21}\, e_2 \epsilon_1 + a_{i22}\, e_2 \epsilon_2 + \dots + a_{i2n}\, e_2 \epsilon_n \\ + & \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ + & a_{in1} e_n \epsilon_1 + a_{in2} e_n \epsilon_2 + \dots + a_{inn} e_n \epsilon_n, \end{aligned}$$

where $e_1 e_2 \dots e_n = \epsilon_1 \epsilon_2 \dots \epsilon_n = 1$], if we write

$$\begin{aligned} A_i &= a_{i1} \epsilon_1 + a_{i2} \epsilon_2 + \dots + a_{in} \epsilon_n, \\ B_i &= b_{i1} \epsilon_1 + b_{i2} \epsilon_2 + \dots + b_{in} \epsilon_n, \end{aligned}$$

it is clear that the value of our cubic determinant is

$$C = (A_1 e_1 + A_2 e_2 + \dots + A_n e_n)^m\, (B_1 e_1 + B_2 e_2 + \dots + B_n e_n)^{n-m}.$$

Now products of pairs of alternate numbers such as $A_i e_i$ or $B_i e_i$ are what Mr. Spottiswoode* has called concurrent numbers; namely, we have

$$(A_i e_i)^2 = 0 = (B_i e_i)^2,$$

$$(A_i e_i)\, (A_k e_k) = (A_k e_k)\, (A_i e_i).$$

Hence it is clear that the first expression on the right for C reduces to the sum of such terms as

$$m!\ A_p e_p A_q e_q \dots A_r e_r,$$

where $p, q \dots r$ is any m-ad selected from $1, 2 \dots n$.

Similarly, the second expression on the right for C reduces to the sum of such terms as

$$(n-m)!\ B_u e_u B_v e_v \dots B_w e_w,$$

where $u, v \dots w$ is any $(n-m)$-ad selected from $1, 2 \dots n$. The product of these two sums will give C. On multiplying the two sums

* *Proceedings of Mathematical Society*, Vol vii., p. 101.

together, we shall get a series of terms, such as

$$m!\ (n-m)!\ A_p e_p A_q e_q \ldots A_r e_r B_u e_u B_v e_v \ldots B_w e_w ,$$

and in this series all terms vanish where the combinations

$$p, q \ldots r \text{ and } u, v \ldots w$$

contain any suffix in common. They must therefore be complementary combinations selected from $1, 2 \ldots n$. Bringing the order of the suffixes both of the units e and of the alternate numbers A and B to the order $1, 2 \ldots n$, we leave the sign of the term unaltered, and hence

$$C = m!\ (n-m)!\ S ;$$

where S is the sum of the $\dfrac{n!}{m!\ (n-m)!}$ ordinary determinants obtained by substituting for each combination m at a time of the rows of the determinant $B = | b_{ik} |$ the corresponding m rows of the determinant $A = | a_{ik} |$.

If the determinants A and B become identical, each term of the sum S becomes equal to A, and hence

$$C = n!\ A,$$

a result which is well known.

2. In the theorem of Art. 1, let us suppose that the system b_{ik} is the system A_{ik} of first minors of the determinant $| a_{ik} |$. Also let $m = 1$. Then the sum S is the sum of n such determinants as

$$\begin{vmatrix} A_{11} & \ldots\ldots & A_{1n} \\ A_{21} & \ldots\ldots & A_{2n} \\ \ldots & \ldots\ldots & \ldots \\ a_{i1} & \ldots\ldots & a_{in} \\ \ldots & \ldots\ldots & \ldots \\ A_{n1} & \ldots\ldots & A_{nn} \end{vmatrix}$$

Expanding this according to the elements of the i^{th} row, we get, by a well-known theorem regarding the first minors of the reciprocal system,

$$A^{n-2} (a_{i1}^2 + a_{i2}^2 + \ldots + a_{in}^2).$$

Hence in this case the determinant C is given by the equation

$$C = (n-1)\ !\ A^{n-2} \Sigma\, a_{ik}^2 ;$$

the sum being the squares of the elements of the determinant $A = | a_{ik} |$.

3. If in the theorem of Art. 1 we repeat the stratum $| a_{ik} |$ $n-1$ times and the stratum $| A_{ik} |$ only once, the resulting cubic determinant is in like manner $C = (n-1)\ !\ \Sigma\, A_{ik}^2 ;$

the sum being the sum of the squares of first minors of the determinant A.

4. If in the theorem of Art. 1 we replace the stratum b_{ik} by the reciprocal system A_{ik} of a_{ik} and expand each element of the sum S by Laplace's theorem keeping the m rows from the determinant A together, we have, by known theorems regarding minors of reciprocal systems,

$$S = A^{n-m-1} \Sigma p_{ik}^2,$$

where p_{ik} is any minor of $A = | a_{ik} |$ of order m. Hence, if we form the strata of a cubic determinant by repeating the determinant $A = | a_{ik} |$ m times, and the reciprocal system $| A_{ik} |$ $n-m$ times; this cubic determinant is given by

$$C = m\,!\,(n-m)\,!\,A^{n-m-1} \Sigma p_{ik}^2.$$

5. We can, however, obtain another cubic determinant which also, contains as a factor the sum of the squares of all minors of $A = | a_{ik} |$ of order m. Namely, if $\dfrac{n\,!}{m\,!\,(n-m)\,!} = \mu = n_m$, and if

$$\begin{array}{cc} p_{11} \cdots p_{1\mu} & q_{11} \cdots q_{1\mu} \\ \cdots\ \cdots\ \cdots\ \cdots & \cdots\ \cdots\ \cdots\ \cdots \\ p_{\mu 1} \cdots p_{\mu\mu} & q_{\mu 1} \cdots q_{\mu\mu} \end{array}$$

are the systems of minors of the determinant A, of orders m and $n-m$ respectively. Then, if we form a cubic determinant of order μ by taking $\mu-1$ strata, all identical with q_{ik}, and one stratum p_{ik}; then, by the theorems of Arts. 1 and 2 and by known theorems in the theory of compound determinants,

$$\begin{aligned} C &= (\mu-1)\,!\,A^{\mu-1-(n-1)_{m-1}} \Sigma p_i^2 \\ &= (\mu-1)\,!\;\; A^{(n-1)_{m-1}} \Sigma p_{ik}^2. \end{aligned}$$

The ratio of this cubic determinant to that in Art. 4 is (up to a numerical factor) a power of A.

If we form the strata of a cubic determinant of order μ by repeating the stratum p_{ik} r times, and the stratum q_{ik} $\mu-r$ times, the resulting cubic determinant is in like manner given by

$$C = r\,!\,(\mu-r)\,!\,A^{\mu-r-(n-1)_{m-1}} \Sigma \pi_{ik}^2,$$

where π_{ik} is the system of minors of order r of the determinant $| p_{ik} |$.

6. It is interesting to consider a cubic determinant in which all elements vanish except those situated in the three diagonal planes passing through pairs of opposite edges of the cube in which the elements are arranged.

First, consider the diagonal plane which cuts the strata orthogonally;

then all the elements of the determinant $| a_{ik} |$ be taken to form these diagonals, so that the first row is taken to form the principal diagonal of the first stratum, and so on. Then the cubic determinant reduces to the product

$$C = \Pi (a_{i1} e_1 \epsilon_1 + a_{i2} e_2 \epsilon_2 + \ldots + a_{in} e_n \epsilon_n)$$

But such products as $e_i \epsilon_i$ are concurrent numbers, hence

$$C = \left\{ \begin{matrix} a_{11}, & a_{12} \ldots a_{1n} \\ a_{21}, & a_{22} \ldots a_{2n} \\ \ldots & \ldots \ldots \ldots \\ a_{n1}, & a_{n2} \ldots a_{nn} \end{matrix} \right\}.$$

Where the function on the right is formed like a determinant, only all the terms have the positive sign. If we have a cubic determinant in which all the elements vanish except those lying in one of the two diagonal planes of the cube which cut the strata at an angle of 45°, it is clear that in the one case all the elements of each stratum vanish except those of one row, and in the other all vanish except those of one column. For convenience, let us again take the several rows or columns to be those of the determinant $| a_{ik} |$. Then, with the notation of Art. 1, it is clear that in the first case the cubic determinant reduces to

$$C = \Pi (A_i e_i)$$
$$= | a_{ik} | ,$$

and in the second case to

$$C = \Pi (a_{1i} e_1 + a_{2i} e_2 + \ldots + a_{ni} e_n) \epsilon_i$$
$$= | a_{ik} | .$$

Thus, in each of these two cases, the cubic determinant reduces to the ordinary determinant which can be formed from the non-vanishing elements of the cubic determinant.

7. A considerable number of cubic determinants can be obtained by forming their strata from an ordinary determinant. As an example, if we form a cubic determinant from the ordinary determinant $| a_{ik} |$ by multiplying the i^{th} row by $1 + h$ $(i = 1, 2 \ldots n)$, and using this as the i^{th} stratum of the cubic determinant, then the cubic determinant is given by $C = \Pi (A_1 e_1 + A_2 e_2 + \ldots + A_n e_n + h A_i e_i)$.

Or, if we write for brevity

$$X = A_1 e_1 + A_2 e_2 + \ldots + A_n e_n ,$$
$$c_i = A_i e_i ,$$

then
$$C = (X+hc_1)(X+hc_2)\dots(X+hc_n)$$
$$= X^n + X^{n-1}h\,(c_1+c_2+\dots+c_n)+\dots$$
$$+X^{n-r}h^r p_r+\dots+h^n c_1c_2\dots c_n;$$

where p_r is the sum of the products of the quantities $c_1, c_2 \dots c_n$ taken r at a time. But, clearly,

$$X^{n-r}p_r = (n-r)!\ \mid a_{ik}\mid \frac{n!}{r!\,(n-r)!}$$
$$= \frac{n!}{r!}A.$$

Hence
$$C = A\,n!\left\{1+\frac{h}{1!}+\frac{h^2}{2!}+\dots+\frac{h^n}{n!}\right\}.$$

If we had formed the strata of the cubic determinant by simply repeating the system a_{ik} n times, the value of the resulting cubic determinant would have been

$$C' = An!$$

Hence we see that, whatever the system $\mid a_{ik}\mid$ may be,

$$\frac{C}{C'} = 1+\frac{h}{1!}+\frac{h^2}{2!}+\dots+\frac{h^n}{n!}.$$

$\left\{\text{If we increase } n \text{ indefinitely } \frac{C}{C'} = e^h.\right\}$

8. If we form the i^{th} stratum of a cubic determinant by adding the quantities $c_1, c_2 \dots c_n$ to the i^{th} row of the determinant $\mid a_{ik}\mid$, the resulting cubic determinant is given by

$$C = \Pi\,(A_1e_1+A_2e_2+\dots+A_ne_n+ce_i),$$

where
$$c = c_1\epsilon_1+c_2\epsilon_2+\dots+c_n\epsilon_n.$$

By effecting the multiplication we get

$$C = n!\,A+(n-1)!\,\Sigma\begin{vmatrix} a_{11} & \dots\dots & a_{n1} \\ \dots & \dots & \dots \\ c_1 & \dots\dots & c_n \\ \dots & \dots & \dots \\ a_{n1} & \dots\dots & a_{nn}\end{vmatrix}_p$$

where the suffix p means that we have substituted the row $c_1, c_2 \dots c_n$ for the p^{th} row of the determinant A. The summation extending over all the values of p from 1 to n, it is clear that the coefficient of c in the sum is
$$A_{1r}+A_{2r}+\dots+A_{nr};$$

also that the sum can be written in the form

$$-\begin{vmatrix} a_{11} & \dots & a_{1n}, & 1 \\ \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn}, & 1 \\ c_1 & \dots & c_n & \end{vmatrix}$$

Hence the cubic determinant of order n reduces to an ordinary determinant of order $(n+1)$; viz., we have

$$C = -(n-1)! \begin{vmatrix} a_{11} & \dots & a_{1n}, & 1 \\ \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn}, & 1 \\ c_1 & \dots & c_n, & -n \end{vmatrix}$$

9. If for each element a_{ik} of an ordinary determinant we write $a_{ik}+c$, we get n^2 new determinants a_{ik}. These can be taken to form another determinant $|\ a_{ik}\ |$, whose value we shall now proceed to find.

We have clearly $\qquad a_{ik} = A + cA_{ik}.$

Hence
$$|\ a_{ik}\ | = \begin{vmatrix} A+cA_{11} & \dots & A+cA_{1n} \\ \dots & \dots & \dots \\ A+cA_{n1} & \dots & A+cA_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} 1, & 0 & \dots & 0 \\ 1, & A+cA_{11} & \dots & A+cA_{1n} \\ \dots & \dots & \dots & \dots \\ 1, & A+cA_{n1} & \dots & A+cA_{nn} \end{vmatrix}$$

Or, multiplying the first column by A and subtracting it from each of the other columns,

$$|\ a_{ik}\ | = \begin{vmatrix} 1, & -A & \dots & -A \\ 1, & cA_{11} & \dots & cA_{1n} \\ \dots & \dots & \dots & \dots \\ 1, & cA_{n1} & \dots & cA_{nn} \end{vmatrix}$$

Expanding this determinant, by Cauchy's theorem, according to products of elements in the first row and column, we have

$$|\ a_{ik}\ | = c^n A^{n-1} + c^{n-1} A^{n-1} S,$$

where S is the sum of all the elements a_{ik}.

This theorem has been introduced as the first of a series, of which two more will be proved in the following paragraphs by the use of alternate numbers. It will be useful therefore to give the proof of this theorem by aid of alternate numbers. We have then

$$|\ a_{ik}\ | = \Pi\{c(A_{i1}e_1 + A_{i2}e_2 + \dots + A_{in}e_n) + AE\},$$

where $\qquad E = e_1 + e_2 + \dots + e_n.$

But this product already

$$= c^n A^{n-1} + A c^{n-1} \Sigma \begin{vmatrix} A_{11} & \dots & A_{1n} \\ \dots & \dots & \dots \\ 1 & \dots & 1 \\ \dots & \dots & \dots \\ A_{n1} & \dots & A_{nn} \end{vmatrix}_p$$

the suffix p, as before, meaning that in the determinant $| A_{ik} |$ we have replaced the p^{th} row by the row 1, 1 ... 1. The summation extends to all values of p from 1 to n.

Expanding each of the determinants in this sum according to the elements of the p^{th} row, this expression reduces to the same result as before.

10. If now we have the n quantities c_1, c_2 ... c_n, and if we substitute $c_p + a_{ik}$ for a_{ik} in the determinant A, we get a series of n^3 elements,

$$c_{pik} = A + c_p A_{ik},$$

which we can take to form a cubic determinant C. Representing this determinant by a product of alternate numbers in the usual way, we have

$$\begin{aligned} C = \Pi \{ AEE' + c_p (& A_{11}e_1\epsilon_1 + A_{12}e_1\epsilon_2 + \dots + A_{1n}e_1\epsilon_n \\ & + A_{21}e_2\epsilon_1 + A_{22}e_2\epsilon_2 + \dots + A_{2n}e_2\epsilon_n \\ & + \dots\dots \\ & + A_{n1}e_n\epsilon_1 + A_{n2}e_n\epsilon_2 + \dots + A_{nn}e_n\epsilon_n) \} ; \end{aligned}$$

where

$$E = e_1 + e_2 + \dots + e_n,$$

$$E' = \epsilon_1 + \epsilon_2 + \dots + \epsilon_n.$$

If we write

$$a_i = A_{i1}\epsilon_1 + A_{i2}\epsilon_2 + \dots + A_{in}\epsilon_n,$$

then it is clear that we have

$$\begin{aligned} C &= \Pi \{ AEE' + c_p (a_1e_1 + a_2e_2 + \dots + a_ne_n) \} \\ &= n\,!\, c_1c_2 \dots c_n A^{n-1} + (n-1)\,!\, c_1c_2 \dots c_n A \Sigma \frac{1}{c_p} \begin{vmatrix} A_{11} & \dots & A_{1n} \\ \dots & \dots & \dots \\ 1 & \dots & 1 \\ \dots & \dots & \dots \\ A_{n1} & \dots & A_{nn} \end{vmatrix} \end{aligned}$$

the suffix p having the same meaning as before.

Reducing the determinant, we have

$$C = (n-1)\,!\, A^{n-1} c_1c_2 \dots c_n \left\{ n + \Sigma \frac{a_{p1} + a_{p2} + \dots + a_{pn}}{c_p} \right\} ;$$

the summation extending to all values of p.

11. Let us now take the n^2 quantities

$$\begin{matrix} b_{11} & \ldots\ldots & b_{1n} \\ \ldots & \ldots & \ldots \\ b_{n1} & \ldots\ldots & b_{nn} \end{matrix}$$

and let the determinant formed with them be denoted by B. Then, if for the element a_{ik} of the determinant A we substitute $a_{ik}+b_{pq}$ we get n^4 elements

$$c_{pqik} = A + b_{pq} A_{ik},$$

which we can take to form the elements of a determinant C of the fourth class. Representing this as a product of alternate numbers, we have

$$C = \Pi \{ A E_1 E_2 E_3 + B_p X \} ;$$

where

$$\begin{aligned} E_1 &= e_1 + e_2 + \ldots + e_n, \\ E_2 &= \epsilon_1 + \epsilon_2 + \ldots + \epsilon_n, \\ E_3 &= \eta_1 + \eta_2 + \ldots + \eta_n, \\ B_p &= b_{p1} e_1 + b_{p2} e_2 + \ldots + b_{pn} e_n, \\ X &= a_1 \eta_1 + a_2 \eta_2 + \ldots + a_n \eta_n. \end{aligned}$$

Multiplying out this product, we obtain

$$C = n\,!\, A^{n-1} B + \Sigma X^{n-1} A E_2 E_3 \begin{vmatrix} b_{11} & \ldots\ldots & b_{1n} \\ \ldots & \ldots & \ldots \\ 1 & \ldots\ldots & 1 \\ \ldots & \ldots & \ldots \\ b_{n1} & \ldots\ldots & b_{nn} \end{vmatrix}$$

$$= n\,!\, A^{n-1} B + (n-1)\,!\, A^{n-1} S_1 S_2;$$

where

$$\begin{aligned} S_1 = {} & a_{11} + a_{12} + \ldots + a_{1n} \\ & + a_{21} + a_{22} + \ldots + a_{2n} \\ & + \ldots \quad \ldots \quad \ldots \quad \ldots \\ & + a_{n1} + a_{n2} + \ldots + a_{nn}, \\ S_2 = {} & B_{11} + \ldots + B_{1n} \\ & + \ldots \quad \ldots \quad \ldots \\ & + B_{n1} + \ldots + B_{nn}. \end{aligned}$$

Or, since

$$S_2 = - \begin{vmatrix} b_{11} & \ldots\ldots & b_{1n}, & 1 \\ \ldots & \ldots & \ldots & \\ b_{n1} & \ldots\ldots & b_{nn}, & 1 \\ 1 & \ldots\ldots & 1 & \end{vmatrix}$$

$$A^{n-2} S_1 = - \begin{vmatrix} A_{11} & \ldots\ldots & A_{1n}, & 1 \\ \ldots & \ldots & \ldots & \\ A_{n1} & \ldots\ldots & A_{nn}, & 1 \\ 1 & \ldots\ldots & 1 & \end{vmatrix}$$

We may, if we please, write C in the form

$$C = n!\,A^{n-1}B + (n-1)!\,A \begin{vmatrix} A_{11} & \dots & A_{1n}, & 1 \\ \dots & \dots & \dots & \dots \\ A_{n1} & \dots & A_{nn}, & 1 \\ 1 & \dots & 1 & \end{vmatrix} \begin{vmatrix} b_{11} & \dots & b_{1n}, & 1 \\ \dots & \dots & \dots & \\ b_{n1} & \dots & b_{nn}, & 1 \\ 1 & \dots & 1 & \end{vmatrix}$$

12. The following theorem is clearly involved in Arts. 10 and 11.

If $$X = A_1e_1 + A_2e_2 + \dots + A_ne_n,$$

with the same notation as before, and if

$$B_i = b_{i1}\epsilon_1 + b_{i2}\epsilon_2 + \dots + b_{in}\epsilon_n,$$
$$C_k = c_{1k}e_1 + c_{2k}e_2 + \dots + c_{nk}e_n;$$

then $$X^{n-1}B_iC_k = -(n-1)! \begin{vmatrix} a_{11} & \dots & a_{1n}, & c_{1k} \\ a_{21} & \dots & a_{2n}, & c_{2k} \\ \dots & \dots & \dots & \\ a_{n1} & \dots & a_{nn}, & c_{nk} \\ b_{i1} & \dots & b_{in} & \end{vmatrix}$$

For one term of X^{n-1} is

$$(n-1)!\,A_1e_1 \dots A_{r-1}e_{r-1}A_{r+1}e_{r+1} \dots A_ne_n.$$

Multiplying this into $B_i\,C_k$, we have

$$(n-1)!\,A_1e_1 \dots A_{r-1}e_{r-1}A_{r+1}e_{r+1} \dots A_ne_n\,B_i\,C_k$$
$$= (-1)^{n-r}(n-1)! \begin{vmatrix} a_{11} & \dots & \dots & a_{1n} \\ \dots & \dots & \dots & \\ a_{r-11} & \dots & \dots & a_{r-1n} \\ a_{r+11} & \dots & \dots & a_{r+1n} \\ \dots & \dots & \dots & \\ a_{n1} & \dots & \dots & a_{nn} \\ b_{i1} & \dots & \dots & b_{in} \end{vmatrix} c_{rk}.$$

Summing this for all values of r, we get the product $X^{n-1}\,B_i\,C_k$, by Cauchy's theorem. This gives us at once the theorem in question.

This remarkable theorem can be generalised as follows :—If from the three determinants of order n,

$$A = |\,a_{ik}\,|, \quad B = |\,b_{ik}\,|, \quad C = |\,c_{ik}\,|,$$

we form a determinant of order $n+m$ by bordering the determinant A with m rows of B and m columns of C, then, if

$$A_i = a_{i1}\,\epsilon_1 + \dots + a_{in}\,\epsilon_n,$$
$$B_k = b_{k1}\,\epsilon_1 + \dots + b_{kn}\,\epsilon_n,$$
$$C_r = c_{1r}\,e_1 + \dots + c_{nr}e_n,$$
$$X = A_1\,e_1 + \dots + A_n\,e^n,$$

the determinant obtained by bordering A with the u^{th}, v^{th} ... rows of B and the p^{th}, q^{th} ... columns of C, is equal to

$$\frac{(-1)^m}{(n-m)\,!}\, X^{n-m} B_u B_v \ldots C_p C_q \ldots$$

13. From the expressions in Art. 12, can be deduced with the the greatest facility theorems regarding compound determinants whose elements are bordered determinants such as the above, these determinants being in fact reduced to determinants of alternate numbers. The following theorem is an example of the class of theorems referred to. Taking the cubic determinant of order n

$$A = | a_{ijk} | ,$$

if we border the i^{th} stratum by a row and column from the ordinary determinants $B = | b_{ik} |$, $C = | c_{ik} |$, we can form a new cubic determinant whose elements are of the form

$$P_{irk} = \begin{vmatrix} a_{i11} & \ldots\ldots & a_{i1n}, & c_{1k} \\ \ldots & \ldots & \ldots & \ldots \\ a_{in1} & \ldots\ldots & a_{inn}, & c_{nk} \\ b_{r1} & \ldots\ldots & b_{rn} & \end{vmatrix} .$$

Then it can be proved that this cubic determinant is equal to

$$(-1)^n BC \mid A_{ijk} \mid ;$$

where A_{ijk} is the minor of a_{ijk} in the ordinary determinant formed from the elements of the i^{th} stratum of the given cubic determinant; namely, by Cauchy's theorem,

$$P_{irk} = -\Sigma\, b_{rt}\, c_{sk}\, A_{its},$$

or

$$\begin{aligned} -P_{irk} &= c_{1k}\,(b_{r1}\,A_{i11} + b_{r2}\,A_{i21} + \ldots\ldots) \\ &\quad + c_{2k}\,(b_{r1}\,A_{i12} + b_{r2}\,A_{i22} + \ldots\ldots) \\ &\quad + \ldots\ldots \\ &= c_{1k}\,a_{ir1} + c_2\ a_{ir2} + \ldots\ldots \text{ say.} \end{aligned}$$

Thus

$$\begin{aligned} \mid P_{irk} \mid &= (-1)^n C \mid a_{irk} \mid \\ &= (-1)^n BC \mid A_{irk} \mid \end{aligned}$$

by the rule for multiplying a cubic by an ordinary determinant.

On the Flow of Viscous Liquid in a Pipe or Channel.

By A. G. Greenhill.

[*Read Dec. 8th*, 1881.]

The object of this paper is to point out the analogy between the steady motion of a viscous liquid in a pipe or channel under gravity, and the motion of a frictionless liquid in a cylinder, having the same cross section as the pipe or channel, due to any motion of the cylinder perpendicular to its generating lines.

Mr. J. G. Butcher has established the differential equation

$$\frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + M = 0$$

("Proceedings of the London Mathematical Society," Vol. VIII., p. 117) for the motion of viscous liquid through a straight pipe or in a straight channel, where u is the velocity of the liquid, which is parallel to the axis of the pipe or channel.

Also $u = 0$ at the sides of the pipe or channel, and for an open channel $\frac{du}{dz} = 0$ at the free surface $z = 0$.

With Mr. Butcher's notation, ρ denotes the density of the liquid, μ the coefficient of viscosity, α the inclination of the pipe or channel to the horizon; and then, for a channel, $M = \frac{g\rho \sin \alpha}{\mu}$; for a closed pipe, $M = \frac{g\rho \sin \alpha}{\mu} + \frac{p_0 - p_1}{\mu l}$, where l is the length of the pipe, and p_0, p_1 the pressures at the beginning and end of the pipe.

If we take a cylinder with the same cross section as the pipe, filled with frictionless liquid, originally at rest, and rotate the cylinder about the axis of x, which is parallel to the generating lines of the cylinder, with angular velocity ω, then the instantaneous stream function ψ of the motion is such that

$$\frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} = 0$$

at every point of the interior of the cylinder; and

$$\psi = \tfrac{1}{2}\omega\,(y^2 + z^2)$$

at every point of the surface of the cylinder.

If we put

$$u = \psi - \tfrac{1}{2}\omega\,(y^2 + z^2),$$

then

$$\frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\omega = 0,$$

at every point of the interior, and $u = 0$ at the surface of the cylinder; and then u is the stream function of the motion of the liquid relative to the cylinder, and $u =$ constant is the equation of a relative stream line.

If the frictionless liquid filling the cylinder had been rotating about the axis of x without relative displacement of the liquid particles, that is to say, as if rigid, then any alteration of this angular velocity would cause motion in the liquid relative to the cylinder, and $u =$ const. would be the equation of the relative stream lines, and the motion would be steady, in consequence of the uniform vorticity throughout the liquid.

Consequently, if we put $M = 2\omega$, the analytical determination of u is the same in the two problems of the flow of the viscous liquid through the pipe, and the motion of the frictionless liquid relative to the cylinder having the same cross section as the pipe.

Thus, as examples, when the cross section is—

(i.) the circle $y^2+z^2 = a^2$,

then $$\psi = \text{const.} = \tfrac{1}{2}\omega a^2,$$

and $$u = \text{const.} - \tfrac{1}{2}\omega\,(y^2+z^2)$$
$$= \tfrac{1}{4}M\,(a^2-y^2-z^2);$$

(ii.) the ellipse $\dfrac{y^2}{a^2} + \dfrac{z^2}{b^2} - 1 = 0$,

then $$\psi = \tfrac{1}{2}\omega\,\frac{a^2-b^2}{a^2+b^2}\,(y^2-z^2)$$

and $$u = \frac{\frac{1}{2}M}{\frac{1}{a^2}+\frac{1}{b^2}}\left(1-\frac{y^2}{a^2}-\frac{z^2}{b^2}\right);$$

(iii.) an equilateral triangle,

then $$u = M\alpha\beta\gamma \div h,$$

where h is the altitude of the triangle, and α, β, γ the perpendicular distances of a point from the sides of the triangle.

(iv.) When the cross section is bounded by the hyperbolas

$$\left.\begin{aligned}\sqrt{2}\,(y^2-z^2)+y^2+z^2-a^2 &= 0\\ \sqrt{2}\,(y^2-z^2)-y^2-z^2+b^2 &= 0\end{aligned}\right\},$$

then

$$u = -\tfrac{1}{4}\frac{M}{a^2+b^2}\{\sqrt{2}\,(y^2-z^2)+y^2+z^2-a^2\}\{\sqrt{2}\,(y^2-z^2)-y^2-z^2+b^2\}.$$

Turned through a quarter of a right angle, the hyperbolas become

$$\left.\begin{aligned}2y\,(y-z)-a^2 &= 0\\ 2z\,(y+z)-b^2 &= 0\end{aligned}\right\},$$

and then $$u = \tfrac{1}{4}\frac{M}{a^2+b^2}\{2y\,(y-z)-a^2\}\{2z\,(y+z)-b^2\}.$$

(v.) When the cross section is a sector bounded by $r = a$ and $\theta = \pm\alpha$, where $y = r\cos\theta$, $z = r\sin\theta$, then

$$u = -\tfrac{1}{4}Mr^2 + \tfrac{1}{4}Mr^2\frac{\cos 2\theta}{\cos 2\alpha} + \sum_{i=0}^{i=\infty} A_{2i+1}\left(\frac{r}{a}\right)^{(2i+1)\frac{\pi}{2\alpha}}\cos(2i+1)\frac{\pi\theta}{2\alpha},$$

where

$$A_{2i+1} = \tfrac{1}{2}M\alpha^2(-1)^{i+1}\left\{\frac{1}{(2i+1)\pi-4\alpha} - \frac{2}{(2i+1)\pi} + \frac{1}{(2i+1)\pi+4\alpha}\right\}.$$

(vi.) When the cross section is a rectangle, bounded by $y = \pm a$, $z = \pm b$,

$$u = \tfrac{1}{4}M(a^2+b^2-y^2-z^2) + \frac{8Ma^2}{\pi^3}\sum_{i=0}^{i=\infty}(-1)^i\frac{\cosh(2i+1)\frac{\pi z}{2a}\cos(2i+1)\frac{\pi y}{2a}}{(2i+1)^3\cosh(2i+1)\frac{\pi b}{2a}} + \frac{8Mb^2}{\pi^3}\sum_{i=0}^{i=\infty}(-1)^i\frac{\cosh(2i+1)\frac{\pi y}{2b}\cos(2i+1)\frac{\pi z}{2b}}{(2i+1)^3\cosh(2i+1)\frac{\pi a}{2b}}.$$

Whenever the expression for u satisfies the condition that $\frac{du}{dz} = 0$ for $z = $ const., then u will serve to determine the motion in an open channel, of which $z = $ const. is the free surface.

Thus the preceding value of u will serve for the case of a channel of rectangular section of breadth $2a$ and depth b, $z = 0$ being the free surface.

The delivery of the pipe per unit of time

$$= \iint u\,dy\,dz;$$

and since $\nabla^2 u = -M,$

where $\nabla^2 = \frac{d^2}{dy^2} + \frac{d^2}{dz^2},$

therefore the delivery

$$= -\frac{1}{M}\iint u\,\nabla^2 u\,dy\,dz$$

$$= \frac{1}{M}\iint\left\{\left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2\right\}dy\,dz,$$

by Green's transformation, since $u = 0$ round the boundary.

Now $\psi = \frac{1}{2}\omega(y^2+z^2) + u;$

$$\frac{d\psi}{dy} = \omega y + \frac{du}{dy},$$

$$\frac{d\psi}{dz} = \omega z + \frac{du}{dz};$$

and $\therefore$ $$\iint \left\{ \left(\frac{d\psi}{dy}\right)^2 + \left(\frac{d\psi}{dz}\right)^2 \right\} dy\, dz$$

$$= \omega^2 \iint (y^2 + z^2)\, dy\, dz$$

$$+ 2\omega \iint \left(y \frac{du}{dy} + z \frac{du}{dz} \right) dy\, dz$$

$$+ \iint \left\{ \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2 \right\} dy\, dz$$

and $$2\omega \iint \left(y \frac{du}{dy} + z \frac{du}{dz} \right) dy\, dz$$

$$= -4\omega \iint u\, dy\, dz$$

$$= 2 \iint u \nabla^2 u\, dy\, dz$$

$$= -2 \iint \left\{ \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2 \right\} dy\, dz.$$

Therefore $$\iint \left\{ \left(\frac{d\psi}{dy}\right)^2 + \left(\frac{d\psi}{dz}\right)^2 \right\} dy\, dz$$

$$= \omega^2 \iint (y^2 + z^2)\, dy\, dz$$

$$- \iint \left\{ \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2 \right\} dy\, dz,$$

or $$\iint \left\{ \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2 \right\} dy\, dz$$

$$= \omega^2 Ak^2 - \iint \left\{ \left(\frac{d\psi}{dy}\right)^2 + \left(\frac{d\psi}{dz}\right)^2 \right\} dy\, dz\ ;$$

so that the delivery of the pipe is proportional to the torsional rigidity of a homogeneous elastic cylinder of the same cross section.

Denoting the rigidity of the substance by n and the torsional rigidity of the cylinder by nAk^2 (Thomson and Tait, "Natural Philosophy," §§ 706–710), then the delivery of the pipe or channel will be $\frac{1}{4}MAk^2$.

The preceding solutions have been worked out, independently of the analogy, by Dr. L. Graetz, in the "Zeitschrift für Mathematik und Physik, 1880," "Ueber die Bewegung von Flüssigkeiten in Röhren."

Note on Landen's Theorem. *By* Prof. Cayley.

[*Read Nov. 10th*, 1881.]

Landen's theorem, as given in the paper "An Investigation of a General Theorem for finding the length of any Arc of any Conic Hyperbola by means of two Elliptic Arcs, with some other new and useful Theorems deduced therefrom," *Phil. Trans.*, t. 65 (1775), pp. 283—289, is, as appears by the title, a theorem for finding the length of a hyperbolic arc in terms of the length of two elliptic arcs; this theorem being obtained by means of the following differential identity, viz., if

$$t = gx\sqrt{\frac{m^2-x^2}{m^2-gx^2}}, \quad \text{where} \quad g = \frac{m^2-n^2}{n^2},$$

then $$\sqrt{\frac{m^2-gx^2}{m^2-x^2}}\,dx = \left\{\tfrac{1}{2} + \tfrac{1}{4}\sqrt{\frac{(m-n)^2-t^2}{(m+n)^2-t^2}} + \tfrac{1}{4}\sqrt{\frac{(m+n)^2-t^2}{(m-n)^2-t^2}}\right\} dt,$$

(this is exactly Landen's form, except that he of course writes $\dot{x}$, $\dot{t}$ in place of dx, dt respectively): viz., integrating each side, and interpreting geometrically in a very ingenious and elegant manner the three integrals which present themselves, he arrives at his theorem for the hyperbolic arc; but with this I am not now concerned.

Writing for greater convenience $m = 1$, $n = k'$, and therefore $g = k^2$ (if as usual $k^2+k'^2 = 1$), the transformation is

$$t = k^2x\sqrt{\frac{1-x^2}{1-k^2x^2}},$$

leading to $$\sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx = \left\{\tfrac{1}{2} + \tfrac{1}{4}\sqrt{\frac{(1-k')^2-t^2}{(1+k')^2-t^2}} + \sqrt{\frac{(1+k')^2-t^2}{(1-k')^2-t^2}}\right\} dt.$$

The form in which the transformation is usually employed (see my "Elliptic Functions," pp. 177, 178) is

$$y = (1+k')\,x\sqrt{\frac{1-x^2}{1-k^2x^2}},$$

leading to $$\frac{(1+k')\,dx}{\sqrt{1-x^2}\,.\,\sqrt{1-k^2x^2}} = \frac{dy}{\sqrt{1-y^2}\,.\,\sqrt{1-\lambda^2y^2}},$$

where $$\lambda = \frac{1-k'}{1+k'}.$$

Or, if to identify the two forms we write $y = \dfrac{t}{1-k'}$, and in the last equation introduce t in place of y, the last equation becomes

$$\frac{dx}{\sqrt{1-x^2}\,.\,\sqrt{1-k^2x^2}} = \frac{dt}{\sqrt{(1-k')^2-t^2}\,.\,\sqrt{(1+k')^2-t^2}}.$$

Comparing with Landen's form, in order that the two may be identical, we must have

$$1-k^2x^2 = \left\{ \tfrac{1}{2} + \tfrac{1}{4}\sqrt{\frac{(1-k')^2-t^2}{(1+k')^2-t^2}} + \tfrac{1}{4}\sqrt{\frac{(1+k')^2-t^2}{(1-k')^2-t^2}} \right\} \times \sqrt{(1-k')^2-t^2}\,\sqrt{(1+k')^2-t^2},$$

viz., this is $1-k^2x^2 = \frac{1}{4}\left\{\sqrt{(1-k')^2-t^2}+\sqrt{(1+k')^2-t^2}\right\}^2$,

that is, $1-k^2x^2=\frac{1}{2}\left\{1+k'^2-t^2+\sqrt{(1-k')^2-t^2}\,.\,(1+k')^2-t^2}\right\}$,

where the function under the radical sign is

$$(1-k'^2)^2-2\,(1+k'^2)\,t^2+t^4\,(=T \text{ suppose});$$

and this must consequently be a form of the original integral equation

$$t = k^2x\sqrt{\frac{1-x^2}{1-k^2x^2}}.$$

In fact, squaring and solving in regard to x^2, with assumed sign of the radical, we have

$$x^2 = \frac{k^2+t^2-\sqrt{T}}{2k^2},$$

(corresponding to an equation given by Landen). And we thence have

$$1-k^2x^2 = \tfrac{1}{2}\{2-k^2-t^2+\sqrt{T}\},\ = \tfrac{1}{2}\{1+k'^2-t^2+\sqrt{T}\},$$

which is the required expression for $1-k^2x^2$.

The trigonometrical form $\sin(2\phi'-\phi)=c\sin\phi$ of the relation between y and x does not occur in Landen; it is employed by Legendre, I believe, in an early paper, *Mém. de l'Acad. de Paris*, 1786, and in the *Exercices*, 1811, and also in the *Traité des Fonctions Elliptiques*, 1825, and by means of it he obtains an expression for the arc of a hyperbola in terms of two elliptic functions, $E(c, \phi)$, $E(c', \phi')$, showing that the arc of the hyperbola is expressible by means of two elliptic arcs,—this, he observes, "est le beau théorème dont Landen a enrichi la géométrie." We have, then, (1828), Jacobi's proof, by two fixed circles, of the addition-theorem (see my "Elliptic Functions," p. 28), and the application of this (p. 30) to Landen's theorem is also due to Jacobi [see the "Extrait d'une lettre adressée à M. Hermite," *Crelle*, t. 32 (1846), pp. 176—181]; the connection of the demonstrations, by regarding the point (which is alone necessary for Landen's theorem) as the limit of the smaller circle in the figure for the addition-theorem is due to Durège (see his "Theorie der Elliptischen Functionen," Leipzig, 1861, pp. 168, *et. seq.*)

Note on Professor Cayley's Paper. *By* John Griffiths, M.A.*

With reference to Prof. Cayley's very interesting account of Landen's theorem, the following transformation may be noticed; viz., if

$$\sqrt{1-e^2x^2}+e\sqrt{1-x^2}=\sqrt{(1+e)^2-\frac{4ey^2}{(1+e)^2}} \quad \ldots\ldots\ldots\ldots\ldots(1),$$

then

$$\left\{\sqrt{\frac{1-e^2x^2}{1-x^2}}+e^2\sqrt{\frac{1-x^2}{1-e^2x^2}}+2e\right\}dx=2\sqrt{\frac{(1+e)^2-\frac{4ey^2}{(1+e)^2}}{(1+e)^2-y^2}}\,dy \quad \ldots(2).$$

[This gives the theorem in a geometrical form at once, e being the eccentricity of an ellipse whose semi-axes are 1, $\sqrt{1-e^2}$; x and y the respective abscissæ of corresponding points of this ellipse and a second ellipse of semi-axes $1+e$, $1-e$.]

The principal steps in the above transformation are these:—

(1) is equivalent to

$$x^2=\frac{4y^2}{(1+e)^2}\cdot\frac{1-\frac{y^2}{(1+e)^2}}{(1+e)^2-\frac{4ey^2}{(1+e)^2}} \quad \ldots\ldots\ldots\ldots\ldots(3),$$

or

$$\sqrt{1-x^2\,.\,1-e^2x^2}=\frac{(1+e)^2-2y^2+\frac{4ey^4}{(1+e)^4}}{(1+e)^2-\frac{4ey^2}{(1+e)^2}} \quad \ldots\ldots\ldots\ldots\ldots(4),$$

so that we have

$$\frac{dx}{\sqrt{1-x^2\,.\,1-e^2x^2}}=\frac{2\,dy}{\sqrt{(1+e)^2-y^2\,.\,(1+e)^2-\frac{4ey^2}{(1+e)^2}}} \quad \ldots\ldots(5).$$

The left-hand side of (2) may be written

$$\frac{(\sqrt{1-e^2x^2}+e\sqrt{1-x^2})^2}{\sqrt{1-x^2\,.\,1-e^2x^2}}\,dx,$$

and (2) follows at once from (1) and (5).

The above is identical with the results arrived at in my Notes on Landen's theorem, published in the *Proceedings* of the London Mathematical Society, Vol. xi., No. 161.

In fact, after some slight reductions, (1) can be deduced from the equation $p'=v-\lambda$ given in that paper, by putting $v=\sqrt{1-e^2x^2}$, $\lambda=e\sqrt{1-x^2}$, $\mu=ex$.

[* Inserted here by Professor Cayley's permission.]

There seems to be another transformation by means of the equation $t'' = \frac{\mu v}{\lambda}$ (see paper as quoted), but I have not examined this very fully.

The trigonometrical equation $\sin(2\phi' - \phi) = e \sin\phi$ leads to the same results, if we write either

$$\sqrt{1 - e^2 \sin^2\phi} + e\cos\phi = (1+e)\cos\theta',$$

or

$$\sqrt{1 - e^2 \sin^2\phi} - e\cos\phi = (1-e)\sec\theta',$$

where $(1+e)\sin\theta' = 2\sqrt{e}\sin\phi'$, and $\sin\phi = x$, $(1+e)\sin\phi' = y$;

(5) then is simply $F(e, \phi) = \frac{2}{1+e} F(e_1, \phi')$,

e_1^2 being equal to $\frac{4e}{(1+e)^2}$.

To complete the transformation (2) from a geometrical point of view, we may write $x\sqrt{\frac{1-e^2x^2}{1-x^2}} = t'' = e'^2 z \sqrt{\frac{z^2-e^2}{e'^2z^2-e^2}}$, where $ee' = 1$ and z is the abscissa of the point on the hyperbola (transverse axis $2e$, and eccentricity $\frac{1}{e}$), which corresponds to the abscissæ x and y.

This gives $e^2\sqrt{\frac{1-x^2}{1-e^2x^2}}\,dx = dt'' - \sqrt{\frac{e'^2z^2-e^2}{z^2-e^2}}\,dz$, and we can verify as follows:—viz., if $x = \sin\phi$, $z = e\,\mathrm{cosec}\,\psi$, then

$$\tan\phi\sqrt{1-e^2\sin^2\phi} = t'' = \frac{\cot\psi}{\sqrt{1-e^2\sin^2\psi}},$$

and $$\frac{e^2\cos^2\phi\,d\phi}{\sqrt{1-e^2\sin^2\phi}} = d\{\tan\phi\sqrt{1-e^2\sin^2\phi}\} + \frac{\sqrt{1-e^2\sin^2\psi}}{\sin^2\psi}\,d\psi;$$

i.e., $\sqrt{1-e^2\tan\phi} = \cot\psi$ and $\frac{(1-e^2)\,d\phi}{\cos^2\phi\sqrt{1-e^2\sin^2\phi}} = -\frac{\sqrt{1-e^2\sin^2\psi}}{\sin^2\psi}\,d\psi$.

Besides the transformation (2), we have also

$$\left\{e^2\sqrt{\frac{1-x^2}{1-e^2x^2}} + \sqrt{\frac{1-e^2x^2}{1-x^2}} - 2e\right\}dx = \frac{\{\sqrt{1-e^2x^2} - e\sqrt{1-x^2}\}^2\,dx}{\sqrt{1-x^2}\,.\,1-e^2x^2}$$

$$= \frac{(1-e^2)^2}{(1+e)^2 - \frac{4ey^2}{(1+e)^2}} \cdot \frac{2\,dy}{\sqrt{(1+e)^2-y^2\,.\,(1+e)^2 - \frac{4ey^2}{(1+e)^2}}}$$

$$= \frac{2(1-e^2)^2\,dy}{\{(1+e)^2-y^2\}^{\frac{1}{2}}\left\{(1+e)^2 - \frac{4ey^2}{(1+e)^2}\right\}^{\frac{1}{2}}}.$$

Hence, by addition, $\left\{ e^2 \sqrt{\frac{1-x^2}{1-e^2x^2}} + \sqrt{\frac{1-e^2x^2}{1-x^2}} \right\} dx$

$$= \left\{ \left(\frac{(1+e)^2 - \frac{4ey^2}{(1+e)^2}}{(1+e)^2 - y^2} \right)^{\frac{1}{2}} + \frac{(1-e^2)^2}{\{(1+e)^2-y^2\}^{\frac{1}{2}} \left\{ (1+e)^2 - \frac{4ey^2}{(1+e)^2} \right\}^{\frac{3}{2}}} \right\} dy.$$

On the Oscillations of a Viscous Spheroid. By H. Lamb, M.A.

[*Read Nov. 10th*, 1881.]

The chief problem discussed in this paper is the determination of the effect of viscosity on the gravitational oscillations of a spheroidal mass of liquid, but the solution can be readily adapted to cases where the tendency to the spherical form is due to other causes; for instance, it applies to the vibrations of a globule of water under the influence of the tension of the superficial film. I have put together, in Part I., some preliminary results of analysis; the application to the problem in hand then follows in Part II. The results of Part I. have an interest extending beyond the present application; they are useful, for instance, in the theory of the vibrations of an elastic sphere, and in the theory of the induction of electric currents in a spherical conductor.

I.

I propose to investigate the general solution of the following system of equations, $(\nabla^2+h^2)\,u = 0$, $(\nabla^2+h^2)\,v = 0$, $(\nabla^2+h^2)\,w = 0$.....(1),

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0 \quad \text{.....(2),}$$

subject to the condition of finiteness at the origin of coordinates. The symbol ∇^2 here stands for Laplace's operator $\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}$, and h is a constant.

Let us begin with the case $h = 0$. The functions u, v, w can then be expanded in series of solid harmonics, and it is plain that the terms of degree n in these expansions, say u_n, v_n, w_n, must separately satisfy (2). The equations $\nabla^2 u_n = 0$, $\nabla^2 v_n = 0$, $\nabla^2 w_n = 0$ may therefore be put in the forms

$$\frac{d}{dy}\left(\frac{dv_n}{dx} - \frac{du_n}{dy}\right) = \frac{d}{dz}\left(\frac{du_n}{dz} - \frac{dw_n}{dx}\right), \text{ \&c., \&c.....(3).}$$

Hence the expression

$$\left(\frac{dw_n}{dy}-\frac{dv_n}{dz}\right)dx+(\ldots)\,dy+(\ldots)\,dz \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

is an exact differential, say it $=d\chi_n$. This gives

$$\frac{dw_n}{dy}-\frac{dv_n}{dz}=\frac{d\chi_n}{dx},\quad \frac{du_n}{dz}-\frac{dw_n}{dx}=\frac{d\chi_n}{dy},\quad \frac{dv_n}{dx}-\frac{du_n}{dy}=\frac{d\chi_n}{dz} \quad \ldots\ldots\ldots(5).$$

We notice that $\nabla^2\chi_n=0$, so that χ_n must be a solid harmonic, of degree n. From (5) we also obtain

$$z\frac{d\chi_n}{dy}-y\frac{d\chi_n}{dz}=x\frac{du_n}{dx}+y\frac{du_n}{dy}+z\frac{du_n}{dz}+u_n-\frac{d}{dx}(xu_n+yv_n+zw_n) \ldots(6),$$

with two similar equations. Now it appears, from (1) and (2), with $h=0$, that

$$\nabla^2(xu_n+yv_n+zw_n)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(7),$$

so that we write

$$xu_n+yv_n+zw_n=\phi_{n+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots(8),$$

a solid harmonic of degree $n+1$. Hence (6) may be written

$$(n+1)\,u_n=\frac{d\phi_{n+1}}{dx}+z\frac{d\chi_n}{dy}-y\frac{d\chi_n}{dz}, \text{ \&c., \&c} \quad \ldots\ldots\ldots\ldots\ldots(9).$$

The factor $n+1$ may be dropped without loss of generality, so that we finally obtain, as the complete solution of the proposed system of equations, in the particular case of $h=0$,

$$u=\Sigma\left(\frac{d\phi_{n+1}}{dx}+z\frac{d\chi_n}{dy}-y\frac{d\chi_n}{dz}\right), \text{ \&c., \&c} \quad \ldots\ldots\ldots\ldots\ldots(10),*$$

where ϕ_n, χ_n stand for solid harmonics of degree n.

Before proceeding to the general case, it is convenient to exhibit the known solution of the equation

$$(\nabla^2+h^2)\,u=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(11)$$

in the following form. Whatever the value of u, it can always be expanded in a series of terms of the form R_nS_n, where S_n is a surface harmonic of the n^{th} order, and R_n is a function of r or $(x^2+y^2+z^2)^{\frac{1}{2}}$, only. If we take any such term, and perform ∇^2+h^2 upon it, it is easily seen that we obtain S_n multiplied by a function of r. Hence these terms must satisfy (11) singly. If, then, we suppose R_nS_n expanded in rising powers of h^2, thus

$$R_nS_n=a_0-h^2a_1+h^4a_2-h^6a_3+\text{\&c.} \quad \ldots\ldots\ldots\ldots\ldots(12),$$

we find, on substituting in (11), and equating separately to zero the

* Compare Borchardt's theorem, quoted by Mr. T. Craig, *Phil. Mag.*, Nov. 1880.

coefficients of the various powers of k^2,

$$\nabla^2 a_0 = 0, \quad \nabla^2 a_1 = a_0, \quad \nabla^2 a_2 = a_1, \quad \&c. \quad \ldots\ldots\ldots\ldots (13).$$

The solution of the first of these, subject to the condition of finiteness at the origin, is

$$a_0 = \phi_n \ldots\ldots\ldots\ldots (14),$$

where ϕ_n is any solid harmonic of positive degree n; and the solutions of the remaining equations of the system are obtained by repeated application of the known formula

$$\nabla^2 (r^m \phi_n) = m(2n+m+1) r^{m-2} \phi_n \ldots\ldots\ldots\ldots (15).$$

We thus find $a_1 = \dfrac{r^2}{2.2n+3} \phi_n$, $a_2 = \dfrac{r^4}{2.4.2n+3.2n+5} \phi_n$, &c....(16).

The complete solution of (13) would involve the addition, at each stage of this process, of an arbitrary solid harmonic of order n; but, as each term so introduced would form the starting point of a series of the form (16), it is plain that no loss of generality is incurred by the omission of these terms. We thus obtain, as the complete solution of (10),

$$u = \Sigma \psi_n \phi_n \ldots\ldots\ldots\ldots\ldots\ldots (17),$$

where

$$\psi_n = 1 - \frac{k^2 r^2}{2.2n+3} + \frac{k^4 r^4}{2.4.2n+3.2n+5} - \&c. \ldots\ldots\ldots (18).$$

This is, of course, a well-known result.

The following properties of the functions ψ_n will be of frequent use:

$$r \frac{d\psi_{n-1}}{dr} = -\frac{k^2 r^2}{2n+1} \psi_n \ldots\ldots\ldots\ldots (19),$$

$$\psi_n + \frac{1}{2n+1} r \frac{d\psi_n}{dr} = \psi_{n-1} \ldots\ldots\ldots\ldots (20),$$

$$\psi_n - \psi_{n-1} = \frac{k^2 r^2}{2n+1.2n+3} \psi_{n+1} \ldots\ldots\ldots\ldots (21).$$

It is evident that

$$\psi_0 = \frac{\sin hr}{hr} \ldots\ldots\ldots\ldots (22),$$

and it follows, by repeated application of (19), that

$$\psi_n = (-)^n . 1.3.5 \ldots 2n+1 . \left(\frac{1}{z} \frac{d}{dz} \right)^n \frac{\sin z}{z} \ldots\ldots\ldots (23),$$

where z is written for hr.

We can now proceed to the general solution of the proposed system of equations, (1) and (2). We will confine our attention, for the present, to the parts of u, v, w, say u_n, v_n, w_n, which involve surface harmonics of order n; and we will further suppose u_n, v_n, w_n to be

expanded in rising powers of h^2, thus

$$\left.\begin{aligned} u_n &= a_0 - h^2 a_1 + h^4 a_2 - \&c. \\ v_n &= \beta_0 - h^2 \beta_1 + h^4 \beta_2 - \&c. \\ w_n &= \gamma_0 - h^2 \gamma_1 + h^4 \gamma_2 - \&c. \end{aligned}\right\} \quad \ldots\ldots(24).$$

On substituting these values in (1) and (2), we find

$$\nabla^2 a_0 = 0, \ \nabla^2 \beta_0 = 0, \ \nabla^2 \gamma_0 = 0 \quad \ldots\ldots(25),$$

$$\frac{da_0}{dx} + \frac{d\beta_0}{dy} + \frac{d\gamma_0}{dz} = 0 \quad \ldots\ldots(26),$$

$$\nabla^2 a_1 = a_0, \ \nabla^2 \beta_1 = \beta_0, \ \nabla^2 \gamma_1 = \gamma_0 \quad \ldots\ldots(27),$$

$$\frac{da_1}{dx} + \frac{d\beta_1}{dy} + \frac{d\gamma_1}{dz} = 0 \quad \ldots\ldots(28),$$

and so on. The solution of the system (25) and (26) has already been proved to be $a_0 = \frac{d\phi_{n+1}}{dx} + z\frac{d\chi_n}{dy} - y\frac{d\chi_n}{dz}$, &c., &c. .....(29).

The equations (27) are then satisfied, in virtue of (15), by

$$a_1 = \frac{r^2}{2 \cdot 2n+3} a_0, \quad \beta_1 = \frac{r^2}{2 \cdot 2n+3} \beta_0, \quad \gamma_1 = \frac{r^2}{2 \cdot 2n+3} \gamma_0 \ldots\ldots(30);$$

but these values do not satisfy (28), for they make

$$\frac{da_1}{dx} + \frac{d\beta_1}{dy} + \frac{d\gamma_1}{dz} = \frac{n+1}{2n+3} \phi_{n+1} \quad \ldots\ldots(31).$$

But equations (27) are still satisfied if we add to the above values of a_1, β_1, γ_1, any arbitrary solid harmonics of order n; and the form of (31) suggests, therefore, the addition of terms of the forms

$$Cr^{2n+5} \frac{d}{dx} \frac{\phi_{n+1}}{r^{2n+3}}, \quad Cr^{2n+5} \frac{d}{dy} \frac{\phi_{n+1}}{r^{2n+3}}, \quad Cr^{2n+5} \frac{d}{dz} \frac{\phi_{n+1}}{r^{2n+3}} \quad \ldots\ldots(32).$$

We find that (28) will now be satisfied, provided

$$C = \frac{n+1}{n+2} \cdot \frac{1}{2n+3 \cdot 2n+5} \quad \ldots\ldots(33).$$

Hence a solution of the system (27) and (28) is given by

$$a_1 = \frac{r^2}{2 \cdot 2n+3} a_0 + \frac{n+1}{n+2} \cdot \frac{r^{2n+5}}{2n+3 \cdot 2n+5} \frac{d}{dx} \frac{\phi_{n+1}}{r^{2n+3}}, \ \&c., \ \&c. \ \ldots(34).$$

The *complete* solution of the system in question would involve the addition of terms of the same form as a_0, β_0, γ_0; these are omitted for a reason already indicated.

The values of a_2, β_2, γ_2 can now be written down by means of (15),

viz., we have

$$\alpha_2 = \frac{r^4}{2.4.2n+3.2n+5}\alpha_0 + \frac{n+1}{n+2}\cdot\frac{1}{2n+3.2n+5}\cdot\frac{r^{2n+7}}{2.2n+7}\frac{d}{dx}\frac{\phi_{n+1}}{r^{2n+3}}$$

&c., &c....(35),

and it appears that these values make

$$\frac{d\alpha_2}{dx} + \frac{d\beta_2}{dy} + \frac{d\gamma_2}{dz} = 0 \quad \text{.....(36),}$$

so that no additional terms are necessary. The further course of the investigation is now apparent. Collecting our results, we find, as the complete solution of the equations (1) and (2), subject to the condition of finiteness at the origin,*

$$u = \Sigma\left\{\psi_n\left(\frac{d\phi_{n+1}}{dx} + y\frac{d\chi_n}{dz} - z\frac{d\chi_n}{dy}\right)\right.$$
$$\left. -\frac{n+1}{n+2}\cdot\frac{h^2}{2n+3.2n+5}\psi_{n+2}r^{2n+5}\frac{d}{dx}\frac{\phi_{n+1}}{r^{2n+3}}\right\}, \text{ &c., &c....(37).}$$

It appears, then, that the solutions of the proposed system of equations are of two distinct types, which may be thus written—

First type, $$u = \psi_n\left(y\frac{d\chi_n}{dz} - z\frac{d\chi_n}{dy}\right), \text{ &c., &c. .....(38);}$$

Second type,

$$u = \psi_{n-1}\frac{d\phi_n}{dx} - \frac{n}{n+1}\cdot\frac{h^2r^2}{2n+1.2n+3}\psi_{n+1}\cdot r^{2n+1}\frac{d}{dx}\frac{\phi_n}{r^{2n+1}}, \text{ &c., &c....(39).}$$

The solutions of the second type may also be written in the form

$$u = \psi_{n-1}\frac{d\phi_n}{dx} - \frac{n}{n+1}(\psi_n - \psi_{n-1})r^{2n+1}\frac{d}{dx}\frac{\phi_n}{r^{2n+1}}, \text{ &c., &c.....(40).}$$

We may remark that the solutions of the first type make

$$xu + yv + zw = 0 \quad \text{.....(41),}$$

and those of the second type

$$xu + yv + zw = n\psi_n\phi_n \quad \text{.....(42).}$$

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* This restriction may be removed if the summation Σ in (37) be supposed to include negative as well as positive values of n. The suffixes attached to the symbols ϕ_n, χ_n must then be taken to denote the *algebraical degree*, as distinguished from the *order*, of these harmonics.

II.

The equations of small motion of a viscous incompressible fluid are

$$\left.\begin{aligned} \rho\frac{du}{dt} &= \mu\nabla^2 u+\rho\frac{dV}{dx}-\frac{dp}{dx}\\ \rho\frac{dv}{dt} &= \mu\nabla^2 v+\rho\frac{dV}{dy}-\frac{dp}{dy}\\ \rho\frac{dw}{dt} &= \mu\nabla^2 w+\rho\frac{dV}{dz}-\frac{dp}{dz} \end{aligned}\right\} \quad \ldots\ldots(43),$$

and

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0 \quad \ldots\ldots(44),$$

where u, v, w are the component velocities at the point (x, y, z), ρ is the density, μ the coefficient of viscosity, V the gravitation potential, p the pressure. The components of stress at the point (x, y, z) of the fluid are given by the formulæ

$$\left.\begin{aligned} p_{xx} &= -p+2\mu\frac{du}{dx}, \text{ \&c., \&c.}\\ p_{yz} &= \mu\left(\frac{dw}{dy}+\frac{dv}{dz}\right), \text{ \&c., \&c.} \end{aligned}\right\} \quad \ldots\ldots(45),$$

where p_{xx} denotes the tractional stress parallel to x across a plane perpendicular to x, p_{yz} the stress parallel to y across a plane perpendicular to z, &c. Let us apply these formulæ to the small oscillations of a liquid mass about the spherical form, and let the origin be taken at the centre of mass. The conditions to be satisfied at the boundary are that the stress there is to be wholly normal, and constant in amount. Hence we must have, at the boundary,

$$\left.\begin{aligned} xp_{xx}+yp_{xy}+zp_{xz} &= -Px\\ xp_{yx}+yp_{yy}+zp_{yz} &= -Py\\ xp_{zx}+yp_{zy}+zp_{zz} &= -Pz \end{aligned}\right\} \quad \ldots\ldots(46),$$

where P denotes the uniform external pressure. If we substitute from (45), these become

$$2x\frac{du}{dx}+y\left(\frac{du}{dy}+\frac{dv}{dx}\right)+z\left(\frac{du}{dz}+\frac{dw}{dx}\right)=\frac{p-P}{\mu}\,x, \text{ \&c., \&c.},$$

or

$$\left(r\frac{d}{dr}-1\right)u+\frac{d}{dx}(xu+yv+zw)=\frac{p-P}{\mu}\,x, \text{ \&c., \&c.} \quad \ldots\ldots(47),$$

where $\dfrac{d}{dr}$ denotes a differentiation along the radius vector r.

Let us now write $\nu=\mu/\rho$, and

$$\varpi=\frac{p}{\rho}-V \quad \ldots\ldots(48).$$

The equations (43) then become

$$\frac{du}{dt} = \nu\nabla^2 u - \frac{d\varpi}{dx}, \text{ \&c., \&c.} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (49).$$

We now assume that u, v, w, ϖ all vary as $e^{-\alpha t}$; if we omit the exponential factor for shortness, the last written equations become

$$(\nabla^2 + h^2)\, u = \frac{1}{\nu}\frac{d\varpi}{dx}, \text{ \&c., \&c.} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (50),$$

where

$$h^2 = \frac{\alpha}{\nu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (51).$$

It appears from (50) that we must have $\nabla^2\varpi = 0$, so that ϖ can be expanded in the form $\Sigma\varpi_n$, where ϖ_n is a solid harmonic of order n. The complete solution of (50) can now be written down at once; viz., we have

$$u = \frac{1}{\alpha}\frac{d\varpi}{dx} + \text{terms of the types (38) and (39).}$$

The terms of the form (38) represent motions which are everywhere perpendicular to the radius vector, and which are therefore unaffected by gravitation. It is clear, then, that the solutions of the two types are quite independent of one another.

The solutions of the first type may be very briefly dismissed. The equations (50) and (44) are satisfied by $\varpi = 0$ and

$$u = \psi_n\left(y\frac{d\chi_n}{dz} - z\frac{d\chi_n}{dy}\right), \text{ \&c., \&c.} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (38);$$

and, substituting in (47), we find

$$\left[r\frac{d\psi_n}{dr} + (n-1)\,\psi_n\right] = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (52),$$

since, there being no radial motion, the surface value of V is constant. The square brackets are used to indicate that in the enclosed expression r is to be put equal to a, the radius vector of the surface. This equation determines the admissible values of h, which are all real, and the corresponding values of α are then given by (51).

We proceed now to the solutions of the second type, and assume

$$u = \frac{1}{\alpha}\frac{d\varpi_n}{dx} + \psi_{n-1}\frac{d}{dx}\frac{r^n}{a^n}T_n - \frac{n}{n+1}(\psi_n - \psi_{n-1})\frac{r^{2n+1}}{a^{2n+1}}\frac{d}{dx}\frac{a^{n+1}}{r^{n+1}}T_n, \text{ \&c., \&c.}$$
$$\ldots\ldots (53),$$

where T_n is a surface harmonic of order n, and a is the mean radius of the spheroid. We shall find, in fact, that the solutions obtained by giving n different values in (53) are all independent. We have now to substitute from (53) in the surface condition (47). We find, at the

surface,

$$\left(r\frac{d}{dr}-1\right)u=\frac{n-2}{a}\frac{d\varpi_n}{dx}+\left(r\frac{d}{dr}+n-2\right)\psi_{n-1}\cdot\frac{d}{dx}\frac{r^n}{a^n}T_n$$
$$-\frac{n}{n+1}\left(r\frac{d}{dr}+n-2\right)(\psi_n-\psi_{n-1})\cdot\frac{d}{dx}\frac{a^{n+1}}{r^{n+1}}T_n \quad \ldots(54),$$

$$\frac{d}{dx}(xu+yv+zw)=\frac{d}{dx}\left(\frac{n}{a}\varpi_n+n\psi_n\cdot\frac{r^n}{a^n}T_n\right)$$
$$=\frac{n}{a}\frac{d\varpi_n}{dx}+n\cdot\frac{1}{r}\frac{d\psi_n}{dr}\cdot x\frac{r^n}{a^n}T_n+n\psi_n\frac{d}{dx}\frac{r^n}{a^n}T_n$$
$$=\frac{n}{a}\frac{d\varpi_n}{dx}+n\psi_{n-1}\frac{d}{dx}\frac{r^n}{a^n}T_n+n(\psi_n-\psi_{n-1})\frac{d}{dx}\frac{a^{n+1}}{r^{n+1}}T_n \quad \ldots\ldots(55).$$

The reductions have been effected by means of (20), and of the known formula of Spherical Harmonics,

$$x\phi_n=\frac{r^2}{2n+1}\left(\frac{d\phi_n}{dx}-r^{2n+1}\frac{d}{dr}\frac{\phi_n}{r^{2n+1}}\right) \quad \ldots\ldots(56).$$

We have also made $r=a$ wherever possible.

Again, if we assume the equation to the surface of the spheroid to be

$$r=a+S_n \quad \ldots\ldots(57),$$

where S_n is a surface harmonic of the n^{th} order, the surface value of V is known to be, if g denote the value of gravity at the surface,

$$V=\text{const.}-\frac{2(n-1)}{2n+1}gS_n$$
$$=\text{const.}-\frac{\beta^2}{n}aS_n \quad \ldots\ldots(58),$$

if
$$\beta^2=\frac{2n\cdot n-1}{2n+1}\frac{g}{a} \quad \ldots\ldots(59).*$$

Hence we have, at the surface,

$$\frac{p-P}{\mu}\cdot x=\frac{1}{\nu}(\varpi_n+V-P)x$$
$$=\frac{1}{\nu}\left(\varpi_n-\frac{\beta^2 a}{n}S_n\right)x$$
$$=\frac{a^2}{\nu}\cdot\frac{1}{2n+1}\left(\frac{d\varpi_n}{dx}-\frac{d}{dx}\frac{\varpi_n a^{2n+1}}{r^{2n+1}}\right)$$
$$-\frac{a^2}{\nu}\cdot\frac{1}{2n+1}\cdot\frac{\beta^2 a}{n}\left(\frac{d}{dx}\frac{r^n}{a^n}S_n-\frac{d}{dx}\frac{a^{n+1}}{r^{n+1}}S_n\right) \quad \ldots\ldots(60).$$

* The introduction of this constant is suggested, of course, by Sir W. Thomson's result that the time of oscillation of a frictionless spheroid is $2\pi/\beta$, where β has the above value.

But, at the surface, we have also the kinematical condition

$$\frac{dS_n}{dt} = -\alpha S_n = \frac{1}{a}(xu+yv+zw)$$

$$= \frac{1}{a}\left(n\frac{\varpi_n}{\alpha}+n\psi_n T_n\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(61).$$

Hence (60) becomes

$$\frac{p-P}{\mu} = \frac{h^2a^2}{2n+1}\left\{\frac{1}{\alpha}\left(1+\frac{\beta^2}{\alpha^2}\right)\left(\frac{d\varpi_n}{dx}-\frac{d}{dx}\frac{\varpi_n a^{2n+1}}{r^{2n+1}}\right)\right.$$

$$\left.+\frac{\beta^2}{\alpha^2}\psi_n\left(\frac{d}{dx}\frac{r^n}{a^n}T_n-\frac{d}{dx}\frac{a^{n+1}}{r^{n+1}}T_n\right)\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(62).$$

Collecting our results, substituting in the three equations (47), and equating separately the surface harmonics of orders $n-1$ and $n+1$, we obtain, after some reductions,

$$\frac{2n-2}{\alpha}\varpi_n+\left[r\frac{d\psi_{n-1}}{dr}+(2n-2)\psi_{n-1}\right]T_n$$

$$= \frac{h^2a^2}{2n+1}\left[\left(1+\frac{\beta^2}{\alpha^2}\right)\frac{\varpi_n}{\alpha}+\frac{\beta^2}{\alpha^2}\psi_n T_n\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots(63),$$

and $$\frac{n}{n+1}\left[\left(r\frac{d}{dr}-3\right)(\psi_n-\psi_{n-1})T_n\right]$$

$$= \frac{h^2a^2}{2n+1}\left[\left(1+\frac{\beta^2}{\alpha^2}\right)\frac{\varpi_n}{\alpha}+\frac{\beta^2}{\alpha^2}\psi_n T_n\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots(64),$$

where the square brackets indicate surface-values.* These may be written

$$\left\{2(n-1)-\frac{h^2a^2}{2n+1}\left(1+\frac{\beta^2}{\alpha^2}\right)\right\}\frac{[\varpi_n]}{\alpha}$$

$$+\left[2(n-1)\psi_{n-1}-\frac{h^2a^2}{2n+1}\left(1+\frac{\beta^2}{\alpha^2}\right)\psi_n\right]T_n = 0 \quad \ldots\ldots\ldots\ldots(65),$$

$$-\frac{h^2a^2}{2n+1}\left(1+\frac{\beta^2}{\alpha^2}\right)\frac{[\varpi_n]}{\alpha}$$

$$+\left[-\frac{h^2a^2}{2n+1}\left(1+\frac{\beta^2}{\alpha^2}\right)+\frac{n\,.\,2n+4}{n+1\,.\,2n+1}\frac{rd\psi_n}{dr}+\frac{h^2a^2}{n+1}\psi_n\right]T_n = 0 \ldots(66).$$

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* Equations equivalent to (63) and (64) may be obtained in a different manner as follows:—first, by multiplying the three equations (47) in order by x, y, z, and adding; and secondly, by replacing the surface harmonics which arise in (47), by the solid harmonics of positive degree which agree with them at the surface, differentiating with respect to x, y, z in order, and adding. The method here indicated has the advantage of showing, analytically, that the solutions obtained by giving n different integral values in (38) and (53) are all independent.

Eliminating ϖ_n and T_n, we obtain, after several reductions,

$$2(n-1)\frac{h^2a^2}{2n+1}\left(1+\frac{\beta^2}{a^2}\right)\frac{[\psi_{n+1}]}{2n+1\,.\,2n+3}$$

$$=\left\{2(n-1)-\frac{h^2a^2}{2n+1}\left(1+\frac{\beta^2}{a^2}\right)\right\}\left[\frac{\psi_n}{n+1}-\frac{n\,.\,2n+4}{n+1\,.\,2n+1\,.\,2n+3}\psi_{n+1}\right] \quad \ldots\ldots(67).$$

This equation to determine the admissible values of ha is readily solved by approximation in the two extreme cases, where the viscosity is very large or very small.

When ν is very great, ha and α are infinitesimal, so that $[\psi_n]=1$, $[\psi_{n+1}]=1$, approximately. Hence, writing for shortness

$$\zeta=\frac{h^2a^2}{2n+1}\frac{\beta^2}{a^2}=\frac{1}{2n+1}\frac{\beta^2a^2}{\nu\alpha} \quad \ldots\ldots(68),$$

(67) becomes

$$\frac{2\,.\,n-1}{2n+1\,.\,2n+3}\zeta=\frac{2(n-1)-\zeta}{n+1}\left(1+\frac{n\,.\,2n+4}{2n+1\,.\,2n+3}\right) \quad \ldots\ldots(69)$$

approximately. Solving for ζ, and substituting in (68), we find

$$\alpha=\frac{n}{2(n+1)^2+1}\cdot\frac{ga}{\nu} \quad \ldots\ldots(70).$$

This is the result given by Mr. G. H. Darwin, at p. 10 of his paper "On the Bodily Tides of Viscous Spheroids, &c."*

When, on the other hand, ν is small,† it is evident from the investigation‡ of Sir W. Thomson, already alluded to, that $\alpha=i\beta$, nearly, so that ha is large. The series (18) is then not a convenient expression for ψ_n; and we have recourse to the formula (23). From this we find that the most important part of $[\psi_n]$ is

$$(-)^n\,.\,1\,.\,3\,.\,5\,\ldots\,(2n+1)\,\frac{\sin\left(ha+n\frac{\pi}{2}\right)}{(ha)^{n+1}} \quad \ldots\ldots(71).$$

It appears that the ratio $[\psi_{n+1}]/[\psi_n]$ is of the order $1/ha$, so that (67) becomes, approximately,

$$2(n-1)-\frac{h^2a^2}{2n+1}\left(1+\frac{\beta^2}{a^2}\right)=0 \quad \ldots\ldots(72).$$

This leads to

$$\frac{\alpha}{i\beta}=1+\frac{n-1\,.\,2n+1}{h^2a^2}$$

$$=1+\frac{n-1\,.\,2n+1}{i\beta}\cdot\frac{\nu}{a^2},$$

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* *Phil. Trans.*, Part I., 1879.

† That is, relatively. The standard of comparison is βa^2.

‡ *Phil. Trans.*, 1863.

or $$\alpha = i\beta + (n-1)(2n+1)\frac{\nu}{a^2} \quad \text{.....(73).}$$

The most remarkable point about this result is the excessively minute extent to which the oscillations of a globe of moderate dimensions are affected by such a degree of viscosity as is ordinarily met with in nature. If we introduce a symbol τ to denote the "modulus of decay" of the oscillations, *i.e.*, the time which must elapse before their amplitude falls to $1/e$ of its original value, we have

$$\tau = \frac{1}{n-1\,.\,2n+1}\frac{a^2}{\nu} \quad \text{.....(74).}$$

For a globe of the size of the earth, and of the same kinematic viscosity as water, we have, on the C. G. S. system, $a = 6{\cdot}37 \times 10^8$, $\nu = {\cdot}014$; and the corresponding value of τ for the oscillation of longest period ($n = 2$) is

$$\tau = 1{\cdot}84 \times 10'' \text{ years.}$$

Even with the value found by Mr. Darwin* for the viscosity of pitch near the freezing temperature, viz., $\mu = 1{\cdot}3 \times 10^8 \times g$, we find, taking $g = 980$, the value $\tau = 150$ hours

as the modulus of decay of the slowest oscillation of a globe of the size of the earth, having the density of water and the viscosity of pitch.

The nature of the modification introduced by viscosity in the character of the motion will appear most clearly if we calculate the components ξ, η, ζ of the angular velocity of the fluid. Substituting the values of u, v, w from (53), we have

$$2\xi = \frac{dv}{dx} - \frac{du}{dy} = -\frac{h^2}{n+1}\psi_n \left(x\frac{d}{dy} - y\frac{d}{dx}\right)\frac{r^n}{a^n}T_n \quad \text{.....(75).}$$

Now it appears, from (23), that ψ_n is of the same order of magnitude as $e^{\pm ihr}/(hr)^{n+1}$. But we have, approximately,

$$h = \sqrt{\frac{\alpha}{\nu}} = \sqrt{\frac{i\beta}{\nu}} = q(1+i) \quad \text{.....(76),}$$

if $q^2 = \beta/2\nu$. Hence ψ_n is of the order $e^{qr}/(qr)^{n+1}$, where q is large. It appears, then, from (66) and (61), that we have, approximately,

$$\frac{h^2a^2}{n+1}[\psi_n]\,T_n = 2(n-1)\frac{[\varpi_n]}{a} \quad \text{.....(77),}$$

and $$\frac{[\varpi_n]}{a} = -\frac{\alpha a S_n}{n} \quad \text{.....(78).}$$

* *Phil. Trans.*, 1879, p. 16.

Hence (75) becomes

$$2\xi = 2\frac{n-1}{n}\frac{a}{a}\cdot\frac{\psi_n}{[\psi_n]}\left(x\frac{d}{dy}-y\frac{d}{dx}\right)\frac{r^n}{a^n}S_n \quad \ldots\ldots\ldots\ldots\ldots(79).$$

Now, keeping only the most important terms,

$$\frac{\psi_n}{[\psi_n]} = \left(\frac{a}{r}\right)^{n+1}\frac{\sin\left(hr+n\frac{\pi}{2}\right)}{\sin\left(ha+n\frac{\pi}{2}\right)}$$

$$= \left(\frac{a}{r}\right)^{n+1}\{\sin q\,(r-a)+i\cos\,(qr-a)\}\,e^{q(r-a)} \quad \ldots\ldots\ldots(80).$$

We see that the vortices are arranged in spherical strata, and that the strength of the vortex motion rapidly diminishes, at the same time rapidly fluctuating in sign, as the depth below the surface increases. I have elsewhere pointed out the analogy between the diffusion of vortex motion in a viscous fluid and the conduction of heat. In the present case, owing to the oscillatory character of the motion, the sign of the vortex motion which is being diffused inwards from the surface is continually being reversed, so that the effect at a moderate depth becomes insensible, just as the fluctuations of temperature at the earth's surface cease to be felt at a depth of a few yards. In connection with this analogy, it is interesting to note that the kinematic viscosity of water ($\nu = \cdot 014$) does not much exceed the value found by Dr. Everett (·01249) for the thermometric conductivity of the Greenwich gravel.

By supposing S_n to be a sectorial harmonic, taking n a very large number, and fixing our attention on a small equatorial region of the spheroidal surface, we fall on the case of straight parallel waves on a plane sheet of water. The wave length λ must be taken $= 2\pi a/n$, so that the modulus of decay is

$$\tau = \frac{\lambda^2}{8\pi^2\nu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(81).$$

This differs somewhat from the result found by Stokes,* according to which the numerical factor in the denominator should be 16. I have verified the formula (81) by an independent calculation of the effect of viscosity on oscillatory waves.

The whole of the preceding investigations can readily be adapted to cases where the tendency to the spherical form is supplied by other forces than gravity. If, for instance, we consider the vibrations of a

* *Camb. Trans.*, t. 9, p. [61], 1850.

liquid globule under the surface-tension of the bounding film, the only modifications required in the foregoing process are, that we must make $V = 0$ throughout, and that in the surface condition (47) we must replace $p - P$ by

$$p - P - T\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \quad \text{.....(82),}$$

where r_1, r_2 are the principal radii of curvature of the surface, and T is the surface-tension. It appears that

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{2}{a} + (n-1)(n+2)\frac{S_n}{a^2} \quad \text{.....(83);*}$$

so that (67) still holds, provided

$$\beta^2 = n(n-1)(n+2)\frac{T}{\rho a^3} \quad \text{.....(84),}$$

i.e., provided $2\pi/\beta$ is the time of a complete oscillation.† The rest of the solution, and the result (74), is then the same as before. This is only what we should expect from the fact that (74) does not involve g.

The modulus of decay of the slowest oscillation ($n = 2$) of a globule of water is, in seconds,

$$\tau = 14{\cdot}3a^2,$$

where the unit of a is the centimeter.

Addition to the preceding Paper.

A simple verification of the principal result given in the paper referred to can be obtained by means of the "Dissipation Function," whose value is given, in the case of an incompressible fluid, by the formula

$$\frac{F}{2\mu} = \left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dy}\right)^2 + \left(\frac{dw}{dz}\right)^2 + \tfrac{1}{2}\left(\frac{dw}{dy} + \frac{dv}{dz}\right)^2 + \tfrac{1}{2}\left(\frac{du}{dz} + \frac{dw}{dx}\right)^2 + \tfrac{1}{2}\left(\frac{dv}{dx} + \frac{du}{dy}\right)^2 \quad \text{.....(1),}$$

the notation being the same as before. It is tolerably obvious *à priori*, and it can be verified by means of formulæ already given, that when the "kinematical viscosity" ν is small compared with βa^2,‡ we may in

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* R. R. Webb, *Messenger of Mathematics*, t. 9, p. 177, 1880.

† Lord Rayleigh, *Proc. R. S.*, t. 29, 1879.

‡ $2\pi/\beta$ is the time of a complete oscillation, and a is the mean radius of the spheroid.

calculating the value of F neglect, for a first approximation, the modifications introduced into the values of u, v, w by viscosity. Now, in the absence of viscosity, the fundamental modes of vibration of a liquid spheroid would be irrotational, and the velocity-potential ϕ corresponding to the n^{th} mode would be of the form

$$\phi = A \frac{r^n}{a^n} S_n \cos \beta t,$$

where S_n is a surface harmonic of the n^{th} order, and A a constant. The corresponding equation to the external surface is

$$r = a + \frac{n}{\beta a} A S_n \sin \beta t \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

as we see from the consideration that at the surface we must have $dr/dt = d\phi/dr$. The value of the kinetic energy T is then given by

$$2T = \rho \iint \phi \frac{d\phi}{dr} \cdot a^2 d\omega = \rho n a A^2 \cos^2 \beta t \cdot \iint S_n^2 \, d\omega,$$

where $d\omega$ denotes an elementary solid angle having its vertex at the centre of the spheroid. From the general theory of the fundamental modes of vibration of dynamical systems, it is plain that the potential energy V must be given by the formula

$$2V = \rho n a A^2 \sin^2 \beta t \cdot \iint S_n^2 \, d\omega \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

and hence that the total energy is

$$W = \tfrac{1}{2} \rho n a A^2 \cdot \iint S_n^2 \, d\omega \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

If we write $u = d\phi/dx$, &c., the formula (1) becomes

$$\frac{F}{2\mu} = \left(\frac{d^2\phi}{dx^2}\right)^2 + \left(\frac{d^2\phi}{dy^2}\right)^2 + \left(\frac{d^2\phi}{dz^2}\right)^2 + 2\left(\frac{d^2\phi}{dy\,dz}\right)^2 + 2\left(\frac{d^2\phi}{dz\,dx}\right)^2 + 2\left(\frac{d^2\phi}{dx\,dy}\right)^2$$
$$\ldots\ldots\ldots (5).$$

If we multiply the right-hand side of this by $dx\,dy\,dz$, and integrate over the volume of a sphere having its centre at the origin, and its radius $= r$, we obtain

$$\tfrac{1}{2} r^2 \iint \frac{d \cdot q^2}{dr} d\omega,^*$$

where q denotes the value of the velocity at the surface of this sphere.

Now
$$r^2 \iint q^2 d\omega = \frac{d}{dr} \iint \phi \frac{d\phi}{dr} \cdot r^2 d\omega,$$

(each side, when multiplied by $\rho\, dr$, being double the kinetic energy of

* *Motion of Fluids*, p. 61.

the fluid contained between two spheres of radii r and $r+dr$,)

$$= \frac{d}{dr} \cdot n \frac{r^{2n+1}}{a^{2n}} A^2 \cos^2 \beta t \cdot \iint S_n^2 \, d\omega$$

$$= n \,.\, 2n+1 \,.\, \frac{r^{2n}}{a^{2n}} A^2 \cos^2 \beta t \,.\, \iint S_n^2 \, d\omega.$$

From this we find

$$\tfrac{1}{2} r^2 \iint \frac{d \,.\, q^2}{dr} d\omega = n \,.\, n-1 \,.\, 2n+1 \,.\, \frac{r^{2n-1}}{a^{2n}} A^2 \cos^2 \beta t \,.\, \iint S_n^2 \, d\omega.$$

Hence, making $r = a$, we find, from (5), that the total rate of dissipation of energy is

$$\frac{2\mu}{a} \,.\, n \,.\, n-1 \,.\, 2n+1 \,.\, A^2 \cos^2 \beta t \,.\, \iint S_n^2 \, d\omega.$$

The *average* rate of dissipation is obtained from this by omitting the factor $\cos^2 \beta t$, and dividing by 2. Hence

$$\frac{dW}{dt} = -\frac{\mu}{a} \,.\, n \,.\, n-1 \,.\, 2n+1 \,.\, A^2 \,.\, \iint S_n^2 \, d\omega.$$

The effect of the viscosity is to gradually diminish the amplitude of the oscillations, which is proportional to A. Hence, substituting the value of W from (4), we find

$$\frac{dA}{dt} + \frac{A}{\tau} = 0,$$

where

$$\tau = \frac{1}{n-1 \,.\, 2n+1} \cdot \frac{a^2}{\nu}.$$

This agrees with the result obtained in the former paper.

The application of the method of the dissipation function to problems of this kind was first made by Stokes,* in investigating the effect of viscosity on ordinary water waves.

In the above investigation a knowledge of the value of β is not required, unless for the purpose of estimating how far the fundamental assumption that ν is small compared with βa^2 is (in any given case) legitimate. To find β we may calculate the potential energy corresponding to any prescribed form of the spheroidal surface. Thus, when the force governing the oscillations is the mutual gravitation of the parts of the spheroid, the potential energy when the surface has the form $r = a + T_n$ (T_n a surface harmonic of order n) may be found as follows. The gravitation potential at the surface of the spheroid

* *Camb. Trans.*, t. ix., p. [61]. There seems to be a slight oversight in Stokes's calculation. The rate of decay of the *kinetic* energy is equated to the dissipation calculated from (1) above. Since the average kinetic energy is half the total energy, the effect of this is to make the rate of decay of the oscillations come out double the true value.

$r = a + xT_n$ (x being a proper fraction) is known to be

$$= \text{const.} - g\left(1 - \frac{3}{2n+1}\right) xT_n,$$

where g is the value of gravity at the surface of the spheroid. Now let the value of x be increased to $x + dx$; this is equivalent to the addition of a superficial film of surface density $\rho\, dx\,.\, T_n$. Hence the increase of potential energy is

$$dV = 2g\rho \frac{n-1}{2n+1} x\, dx \iint T_n^2\,.\, a^2 d\omega.$$

Integrating this with respect to x between the limits 0 and 1, we find for the total potential energy

$$V = g\rho a^2\,.\, \frac{n-1}{2n+1} \iint T_n^2\, d\omega \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

In our case we have, by (2),

$$T_n = \frac{n}{\beta a} AS_n\,.\cos \beta t.$$

Substituting in (6), and comparing with (3), we find

$$\beta^2 = \frac{2n\,.\,n-1}{2n+1} \frac{g}{a}.$$

This agrees with the result obtained by Thomson, *Phil. Trans.*, 1863, p. 610.

On the Covariant Locus of the Vertex of a Pencil of Tangents to a Cubic in Involution. By J. J. Walker, M.A.

[*Read Dec. 8th*, 1881.]

In a recent communication to the Society by Mr. R. A. Roberts, "On Tangents to a Cubic forming a Pencil in Involution," the locus of the vertex has been very ingeniously found in the shape of the twelve cubics (besides the nine harmonic polars of the inflexion-points),

$$\left.\begin{aligned} (1+8m^3)\, x^3 + 8m^3 U &= 0 \\ (1+8m^3)\, y^3 + 8m^3 U &= 0 \\ (1+8m^3)\, z^3 + 8m^3 U &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\left.\begin{aligned} (1+8m^3)\,(x+y+z)^3 - 8\,(m-1)^3\, U &= 0 \\ (1+8m^3)\,(x+\vartheta y+\vartheta^2 z)^3 - 8\,(m-1)^3\, U &= 0 \\ (1+8m^3)\,(x+\vartheta^2 y+\vartheta z)^3 - 8\,(m-1)^3\, U &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$\left.\begin{aligned} (1+8m^3)\,(\vartheta x+y+z)^3 - 8\,(m-\vartheta)^3\, U &= 0 \\ (1+8m^3)\,(x+\vartheta y+z)^3 - 8\,(m-\vartheta)^3\, U &= 0 \\ (1+8m^3)\,(x+y+\vartheta z)^3 - 8\,(m-\vartheta)^3\, U &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$\left.\begin{aligned}(1+8m^3)\,(\vartheta^2x+y+z)^3-8\,(m-\vartheta^2)^3\,U=0\\(1+8m^3)\,(x+\vartheta^2y+z)^3-8\,(m-\vartheta^2)^3\,U=0\\(1+8m^3)\,(x+y+\vartheta^2z)^3-8\,(m-\vartheta^2)^3\,U=0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots(4),$$

the cubic being supposed to be written in one of its four canonical forms, viz., $U\equiv x^3+y^3+z^3+6mxyz$,

and ϑ, ϑ^2 being the imaginary cube roots of unity.

The equation of the nine harmonic polars has been exhibited as the square root of a covariant by Dr. Salmon (*Higher Plane Curves*, Art. 232).

It occurred to me, on becoming acquainted with the singular form in which Mr. R. A. Roberts found the locus above, to ascertain whether this part of the locus was expressible rationally in the covariant form, or whether, like the other part, its square only was so expressible;* but when I had gone so far as to satisfy myself on this head, I thought it might be as well to complete the work, and be able to exhibit the result in its entirety to those mathematicians who take an interest in such investigations.

As a preliminary it may be well to study briefly the character of these twelve cubics; the first for example is, in full,

$$(1+16m^3)\,x^3+8m^3\,(y^3+z^3+6mxyz)\;=\;0,$$

or, writing, $(1+16m^3)^{\frac{1}{3}}x=\xi,$

$$2my=\eta,$$

$$2mz=\zeta,$$

$$\xi^3+\eta^3+\zeta^3+12\,\frac{m^2}{(1+16m^3)^{\frac{1}{3}}}\,\xi\eta\zeta=0,$$

the discriminant of which is the cube of

$$1+\frac{64m^6}{1+16m^3},$$

or of $$\frac{(1+8m^3)^2}{1+16m^3};$$

so that the cubics will be proper curves of that order, without double points, if U has not one; and obviously each has three inflexions in common with U; viz., the first those lying on $x=0$; and so on.

Multiplying the triad (1), and observing that

$$\begin{aligned}(1+8m^3)\,(x^3+y^3+z^3)\qquad&=(1+2m^3)\,U-6mH,\\(1+8m^3)^2\,(y^3z^3+z^3x^3+x^3y^3)&=-\Theta+m^3(2+m^3)\,U^2\\&\qquad-m\,(1+2m^3)\,UH+3m^2H^2,\\(1+8m^3)\,x^3y^3z^3\qquad&=m^2U+H,\end{aligned}$$

* My attention has been drawn to the *à priori* evidence of this in the fact that the product of the triads (1), (2), (3), (4) does not change sign when x and y are interchanged.

where H is the Hessian and Θ Dr. Salmon's covariant of the sixth degree in the variables, and eighth in the coefficients of U, the result is

$$-8m^3\Theta U+81m^6(1+8m^3)U^3-5m^4(1+80m^3)U^2H$$
$$+3m^2(1+8m^3)UH^2+H^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Similarly, the product of the triad (2) will be

$$-8m'^3\Theta'U'+81m'^6(1+8m'^3)U'^3-5m'^4(1+80m'^3)U'^2H'$$
$$+3m'^2(1+8m'^3)U'H'^2+H'^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

the marked letters indicating the values of m, U, H for the coordinates $x'y'z'$ relative to a second canonical triangle, viz.,

$$x'=x+y+z,$$
$$y'=x+\vartheta y+\vartheta^2 z,$$
$$z'=x+\vartheta^2 y+\vartheta z.$$

These substitutions give—

$$m'=\frac{1-m}{1+2m},$$

$$1+8m'^3=9\frac{1+8m^3}{(1+2m)^4},$$

$$U'=\frac{9U}{1+2m},$$

$$H'=-\frac{27H}{(1+2m)^3},$$

$$\Theta'=-\frac{3^7\Theta}{(1+2m)^8};$$

by means of which (6) is expressed in terms of m, U, H, Θ without difficulty.

Also, the product of the triad (3) will be

$$-8m''^3\Theta''U''+81m''^6(1+8m''^3)U''^3-\ldots\ldots+H''^3 \quad \ldots\ldots\ldots(7),$$

where

$$m''=\frac{\vartheta-m}{\vartheta^2+2\vartheta m}=\frac{1-\vartheta^2 m}{\vartheta+2m},$$

$$1+8m''^3=9\vartheta\frac{1+8m^3}{(\vartheta+2m)^4},$$

$$U''=\frac{9\vartheta U}{\vartheta+2m},$$

$$H''=-\frac{27\vartheta^2 H}{(\vartheta+2m)^3},$$

$$\Theta''=-\frac{3^7\vartheta^3\Theta}{(\vartheta+2m)^8}.$$

Lastly, the product of the triad (4) will, obviously, be derived from that of (3) by interchanging ϑ and ϑ^2.

Thus the products of the triads (2), (3), (4), cleared of extraneous factors, are respectively, writing Θ_1 for 8Θ,

$$\begin{aligned}
&(1+2m)\,(1-m)^3\,\Theta_1 U + 27\,(1-m)^6\,(1+8m^3)\,U^3 \\
&\qquad + 5\,(1-m)^4\,\{10\,(1+8m^3) - (1+2m)^4\}\,U^2 H \\
&\qquad + 9\,(1-m)^2\,(1+8m^3)\,UH^2 - (1+2m)^4\,H^3, \\
&(\vartheta+2m)\,(\vartheta-m)^3\,\Theta_1 U + 27\,(\vartheta-m)^6\,\vartheta\,(1+8m^3)\,U^3 \\
&\qquad + 5\,(\vartheta-m)^4\,\{10\vartheta\,(1+8m^3) - (\vartheta+2m)^4\}\,U^2 H \\
&\qquad + 9\,(\vartheta-m)^2\,\vartheta\,(1+8m^3)\,UH^2 - (\vartheta+2m)^4\,H^3, \\
&(\vartheta^2+2m)\,(\vartheta^2-m)^3\,\Theta_1 U + 27\,(\vartheta^2-m)\,\vartheta^2\,(1+8m^3)\,U \\
&\qquad + 5\,(\vartheta^2-m)^4\,\{10\vartheta^2\,(1+8m^3) - (\vartheta^2+2m)^4\}\,U^2 H \\
&\qquad + 9\,(\vartheta^2-m)\,\vartheta^2\,(1+8m^3)\,UH^2 - (\vartheta^2+2m)^4\,H^3;
\end{aligned}$$

and the continued product of these three, which is rational, is finally multiplied by (5)

$$-m^3\Theta_1 U + 81(1+8m^3)m^6U^3 - 5(1+80m^3)m^4U^2H + 3(1+8m^3)m^2UH + H^3.$$

The result is, writing S, T, Δ for $m\,(1-m^3)$, $1-20m^3-8m^6$, $(1+8m^3)^3$, and rejecting the factor $1+8m^3$,

$$\begin{aligned}
&-S^3\Theta_1^4\,U^4 + \{-81S^6TU^3 + 520S^4U^2H + 3S^2TUH^2 + (40S^3+T^2)\,H^3\}\,\Theta_1^3\,U^3 \\
&+ \{-3^7S^5\Delta U^6 + 120\times 81S^4TU^5H \\
&\quad + S^2(-582\times 64S^3 + 243T^2)\,U^4H^2 + 81T\,(40S^3+T^2)\,U^3H^3 \\
&\quad - 45S\,(5\times 64S^3 + 7T^2)\,U^2H^4 + 72S^2TUH^5 - 3\,(128S^3+T^2)\,H^6\}\,\Theta_1^2\,U^2 \\
&+ \{-27^3S^3T\Delta U^9 - 165\times 27^2S^4\Delta U^8H + 243S^2T\,(128S^3+27T^2)\,U^7H^2 \\
&\quad + (3875\times 512S^6 - 138996S^3\Delta + 2187\Delta^2)\,U^6H^3 \\
&\quad + 9ST\,(295\times 64S^3 - 1125\Delta)\,U^5H^4 \\
&\quad + 3S^2(-525\times 512S^3 + 14182\Delta)\,U^4H^5 - 9T(37\times 64S^3 + 153\Delta)\,U^3H^6 \\
&\quad + 12S\,(110\times 64S^3 + 209T^2)\,U^2H^7 \\
&\quad + 9\times 64S^2TUH^8 + (-512S^3+3\Delta)\,H^9\}\,\Theta_1 U \\
&+ 3^{13}S^6\Delta U^{12} + 95\times 3^9S^4T\Delta U^{11}H + 3^7S^2\,(722S^3+27\Delta)\,\Delta U^{10}H^2 \\
&\quad + T\,(10125\times 512S^3 + 45765S^3\Delta + 19863\Delta^2)\,U^9H^3 \\
&\quad + S\,(-6250\times 4096S^6 + 289980S^3\Delta + 10935\Delta^2)\,U^8H^4 \\
&\quad + S^2T\,(-2025\times 512S^3 + 268218\Delta)\,U^7H^5 \\
&\quad + (3750\times 4096S^6 - 348396S^3\Delta + 86133\Delta^2)\,U^6H^6 \\
&\quad + ST\,(1280\times 64S^3 + 19404\Delta)\,U^5H^7 \\
&\quad - 3S^2\,(250\times 4096S^3 + 128473\Delta)\,U^4H^8 \\
&\quad + 9T\,(-112\times 64S^3 + 225\Delta)\,U^3H^9 \\
&\quad + S\,(3200\times 64S^3 - 4170\Delta)\,U^2H^{10} + 1536S^2TUH^{11} - \Delta H^{12} = 0.
\end{aligned}$$

Thursday, January 12th, 1882.

S. ROBERTS, Esq., F.R.S., President, in the Chair.

Dr. G. J. Allman, Queen's College, Galway, and Mrs. Bryant, B.Sc., F.C.P., were elected members, and Mr. G. H. Stuart was admitted into the Society. A vote of thanks was passed to the Norwegian Government for the present of a copy of the Second Edition of Abel's Works.

The following communications were made :—

"The Invariants of a certain Orthogonal Transformation, with special reference to their use in the Theory of the Strains and Stresses of an Elastic Solid :" Mr. W. J. C. Sharp.

" Certain Elliptic Function Formulæ :" Rev. M. M. U. Wilkinson.

"On the Calculation of Symmetric Functions :" Mr. J. Hammond.

" Complete Determination of the Real Foci, and of the Vector Equation, of the Pedal of a given Ellipse (and Parabola), with respect to any proposed Point :" Prof. Wolstenholme.

The following presents were received :—

Carte-de-Visite likeness from Prof. Genese.

" Educational Times," December, 1881, January, 1882.

" Atti della R. Academia dei Lincei—Transunti," Vol. vi., Fasc. 2°.

" Monatsbericht," November, 1881.

" Beiblätter zu den Annalen der Physik und Chemie," Band v., Stücke 11, 12.

" Œuvres complètes de Niels Henrik Abel," Nouvelle édition, Tomes i., ii.; Christiania, 1881.

" Archiv for Mathematik og Naturvidenskab," Femte Bind, Første, Andet, Tredie, Fjerde Hefte, & Sjette Bind, Første, Andet Hefte, 1881.

Complete Determination of the Real Foci, and of the Vector Equation, of the Pedal of a given Ellipse with respect to any proposed Point. By Prof. WOLSTENHOLME.

[*Read January 12th,* 1882.]

The equation of the pedal of the ellipse $a^2y^2 + b^2x^2 = a^2b^2$ with respect to a point O whose coordinates are X, Y, is

$$(x^2+y^2-xX-yY)^2 = a^2(x-X)^2+b^2(y-Y)^2,$$

or, with the origin at O,

$$(x^2+y^2+xX+yY)^2 = a^2x^2+b^2y^2.$$

The point O is a double focus of this curve, and, to determine the single foci, we determine p so that the straight line $x+iy=p$ may be a tangent. Hence the equation (in $y:x$)

$$[p\,(x-iy)+xX+yY]^2=a^2x^2+b^2y^2$$

must have equal roots; or

$$(p+X)^2\,(p+iY)^2=(\overline{p+X}^2-a^2)\,(\overline{p+iY}^2+b^2),$$

$$\frac{(p+X)^2}{a^2}-\frac{(p+iY)^2}{b^2}=1.$$

Restoring $x+iy$ for p, and equating real and impossible parts,

$$\frac{(x+X)^2-y^2}{a^2}-\frac{x^2-(y+Y)^2}{b^2}=1,$$

$$\frac{(x+X)\,y}{a^2}=\frac{(y+Y)\,x}{b^2};$$

or, restoring the origin to the centre, the equations determining the foci are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{(x-X)^2}{b^2}-\frac{(y-Y)^2}{a^2}=1 \quad \text{.....(A)},$$

$$\frac{x\,(y-Y)}{a^2}=\frac{y\,(x-X)}{b^2} \quad \text{.....(B)}.$$

These equations give us two real foci (F_1, F_2) and two impossible foci (of course the anti-points of F_1, F_2); and the equation (B) proves that OF_1. OF_2 are normals at F_1, F_2 to the ellipses drawn through F_1, F_2, similar, concentric, and similarly situated to the given ellipse. Writing this equation in the form

$$\frac{x-X}{\frac{x}{a^2}}=\frac{y-Y}{\frac{y}{b^2}}\equiv k,$$

we shall obtain two real values for k: let these be k_1, k_2, referring respectively to F_1, F_2. By (A), we get

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{k^2}{a^2b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=1;$$

or $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=\dfrac{a^2b^2}{a^2b^2-k^2}$; whence, since $x=\dfrac{X}{1-\dfrac{k}{a^2}}$, $y=\dfrac{Y}{1-\dfrac{k}{b^2}}$,

$$\frac{X^2}{a^2\left(1-\frac{k}{a^2}\right)^2}+\frac{Y^2}{b^2\left(1-\frac{k}{b^2}\right)^2}=\frac{a^2b^2}{a^2b^2-k^2}.$$

This equation is equivalent to

$$\frac{X^2}{a^2-\lambda}+\frac{Y^2}{b^2-\lambda}=1 \quad \text{.....(C),}$$

where $\lambda=\dfrac{-k^2(a^2+b^2)+2ka^2b^2}{a^2b^2-k^2}$; or, λ being determined by (C), we have for k the quadratic

$$k^2(a^2+b^2-\lambda)-2ka^2b^2+\lambda a^2b^2=0,$$

which gives real values for k if $(a^2-\lambda)(b^2-\lambda)>0$. Hence, for the real foci, we must take the *ellipse* through O confocal with the given ellipse; let a', b' be its semiaxes, then

$$k_1=\frac{a^2b^2+ab\sqrt{a^2-\lambda\,.\,b^2-\lambda}}{a^2+b^2-\lambda},\quad k_2=\frac{a^2b^2-ab\sqrt{a^2-\lambda\,.\,b^2-\lambda}}{a^2+b^2-\lambda},$$

and

$$\frac{a^2b^2}{a^2b^2-k_1^2}=\frac{(a^2+b^2-\lambda)^2}{(a^2+b^2-\lambda)^2-(ab+a'b')^2}$$

$$=\frac{(ab+a'b')^2(ab-a'b')^2}{(ab+a'b')^2(ab-a'b')^2-\lambda^2(ab+a'b')^2},$$

since $\lambda(a^2+b^2-\lambda)=a^2b^2-a'^2b'^2$,

$$=\frac{(ab-a'b')^2}{(ab-a'b')^2-\lambda^2}=\frac{(ab-a'b')^2}{(ab-a'b')^2-(a^2-a'^2)(b^2-b'^2)}$$

$$=\frac{(ab-a'b')^2}{(a'b-ab')^2}.$$

Similarly

$$\frac{a^2b^2}{a^2b^2-k_2^2}=\left(\frac{ab+a'b'}{a'b+ab'}\right)^2;$$

or OF_1, OF_2 are normals at F_1, F_2 to the two ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{ab\mp a'b'}{a'b\mp ab'}\right)^2;$$

a', b' being the semiaxes of the ellipse drawn through O confocal with the given ellipse.

Since at least two real normals can be drawn from O to each ellipse, this alone is not a sufficient determination of the foci F_1, F_2. Let (x_1y_1), (x_2y_2) be their coordinates, then

$$x_1+x_2=\frac{X}{1-\frac{k_1}{a^2}}+\frac{X}{1-\frac{k_2}{a^2}}=\frac{a^2X(2a^2-k_1-k_2)}{(a^2-k_1)(a^2-k_2)}.$$

But

$$k_1+k_2=\frac{2a^2b^2}{a^2+b^2-\lambda},\quad 2a^2-k_1-k_2=\frac{2a^2(a^2-\lambda)}{a^2+b^2-\lambda},$$

and

$$(a^2-k_1)(a^2-k_2)=\frac{a^4(a^2+b^2-\lambda)-2a^4b^2+\lambda a^2b^2}{a^2+b^2-\lambda}=\frac{a^2(a^2-b^2)(a^2-\lambda)}{a^2+b^2-\lambda};$$

so that $$\frac{x_1+x_2}{2}=\frac{a^2X}{a^2-b^2},$$

and similarly $$\frac{y_1+y_2}{2}=\frac{b^2Y}{b^2-a^2}.$$

Thus the middle point of F_1F_2 is determined, and may be constructed as follows:—let fall OG, Og perpendiculars on the axes, and draw normals GP, GP', gp, gp'; the straight lines PP', pp' will intersect in the middle point of F_1F_2; and this will completely determine which of the feet of the normals drawn from O are the foci.

To find the vector equation of the pedal, take the point (a, Y) of the pedal, and let its distances from O, F_1, F_2 be r, r_1, r_2. Then $r=X-a$ (of course r is of doubtful sign), and

$$r_1^2=(x_1-a)^2+(y_1-Y)^2=\left(\frac{X}{1-\frac{k}{a^2}}-a\right)^2+\frac{k_1^2\,Y^2}{(b^2-k_1)^2}$$

$$=\left(\frac{a^2(r+a)}{a^2-k_1}-a\right)^2+\frac{k_1^2}{b^2}\left\{\frac{a^2b^2}{a^2b^2-k_1^2}-\frac{a^2(r+a)^2}{(a^2-k_1)^2}\right\}$$

$$=\frac{a^2(ar+k_1)^2}{(a^2-k_1)^2}+\frac{a^2k_1^2}{a^2b^2-k_1^2}-\frac{a^2k_1^2(r+a)^2}{b^2(a^2-k_1)^2}$$

$$=\frac{a^2(a^2b^2-k_1^2)}{b^2(a^2-k_1)^2}\left\{r+\frac{a^2k_1(b^2-k_1)}{a^2b^2-k_1^2}\right\}^2.$$

Thus $$r_1=\frac{a\sqrt{a^2b^2-k_1^2}}{b(a^2-k_1)}\left\{r+\frac{a^2k_1(b^2-k_1)}{a^2b^2-k_1^2}\right\},$$

and, similarly, $$r_2=\frac{a\sqrt{a^2b^2-k_2^2}}{b(a^2-k_2)}\left\{r+\frac{a^2k_2(b^2-k_2)}{a^2b^2-k_2^2}\right\};$$

whence the equation

$$\frac{b}{a}r_1\frac{(a^2-k_1)\sqrt{a^2b^2-k_1^2}}{ak_1(b^2-k_1)}-\frac{b}{a}r_2\frac{(a^2-k_2)\sqrt{a^2b^2-k_2^2}}{ak_2(b^2-k_2)}$$

$$=\frac{r}{a}\left\{\frac{a^2b^2-k_1^2}{k_1(b^2-k_1)}-\frac{a^2b^2-k_2^2}{k_2(b^2-k_2)}\right\}.$$

But $$a^2-k_1=\frac{aa'(aa'-bb')}{a^2+b^2-\lambda},\quad b^2-k_1=\frac{bb'(bb'-aa')}{a^2+b^2-\lambda},$$

$$\sqrt{a^2b^2-k_1^2}=ab\frac{a'b-ab'}{ab-a'b'},\quad k_1=\frac{ab(ab+a'b')}{a^2+b^2-\lambda};$$

whence $$\frac{(a^2-k_1)\sqrt{a^2b^2-k_1^2}}{k_1(b^2-k_1)}=-\frac{ax'}{bb'}\frac{(a'b-ab')(a^2+b^2-\lambda)}{a^2b^2-a'^2b'^2}$$

$$=-\frac{aa'}{bb'}\frac{a'b-ab'}{\lambda}.$$

So also $$\frac{(a^2-k_2)\sqrt{a^2b^2-k_2^2}}{k_2(b^2-k_2)} = \frac{aa'}{bb'}\,\frac{a'b+ab'}{\lambda}.$$

Moreover, $$\frac{a^2-k_1}{k_1(b^2-k_1)} - \frac{a^2-k_2}{k_2(b^2-k_2)} = -\frac{aa'}{bb'}\left(\frac{1}{k_1}+\frac{1}{k_2}\right) = -\frac{2}{\lambda}\,\frac{aa'}{bb'};$$

so that the vector equation may be written

$$(a'b-ab')\,r_1+(a'b+ab')\,r_2 = 2abr,$$

or $$\left(\frac{a'}{a}-\frac{b'}{b}\right)r_1+\left(\frac{a'}{a}+\frac{b'}{b}\right)r_2 = 2r.$$

We know that O is a point (node) on the pedal, and hence that the equation may be written in the form

$$r_1 \,.\, OF_1 \pm r_2 \,.\, OF_2 = nr \,.\, F_1F_2;$$

and we will now determine n.

$$OF_1^2 = (x_1-X)^2+(y_1-Y)^2 = k_1^2\left\{\frac{X^2}{(a^2-k_1)^2}+\frac{Y^2}{(b^2-k_1)^2}\right\}$$

$$= a^2b^2\left(\frac{ab+a'b'}{aa'-bb'}\right)^2\left(\frac{X^2}{a^2a'^2}+\frac{Y^2}{b^2b'^2}\right)$$

$$= \left(\frac{ab}{a^2-b^2}\right)^2(a'b+ab')^2\left(\frac{X^2}{a^2a'^2}+\frac{Y^2}{b^2b'^2}\right);$$

and, similarly, $$OF_2^2 = \left(\frac{ab}{a^2-b^2}\right)^2(a'b-ab')^2\left(\frac{X^2}{a^2a'^2}+\frac{Y^2}{b^2b'^2}\right);$$

also $$(F_1F_2)^2 = (k_1-k_2)^2\left\{\frac{a^4X^2}{(a^2-k_1)^2(a^2-k_2)^2}+\frac{b^4Y^2}{(b^2-k_1)^2(b^2-k_2)^2}\right\}.$$

But $$k_1-k_2 = \frac{2aa'bb'}{a^2+b^2-\lambda},\quad (a^2-k_1)(a^2-k_2) = \frac{a^2a'^2(a^2-b^2)}{a^2+b^2-\lambda},$$

and $$(b^2-k_1)(b^2-k_2) = \frac{b^2b'^2(b^2-a^2)}{a^2+b^2-\lambda};$$

whence $$(F_1F_2)^2 = 4\left(\frac{ab}{a^2-b^2}\right)^2a'^2b'^2\left(\frac{X^2}{a'^4}+\frac{Y^2}{b'^4}\right),$$

and $$\frac{OF_2^2}{(a'b-ab')^2} = \frac{OF_1^2}{(a'b+ab')^2} = \frac{(F_1F_2)^2}{4a'^2b'^2}\,\frac{\dfrac{X^2}{a^2a'^2}+\dfrac{Y^2}{b^2b'^2}}{\dfrac{X^2}{a'^4}+\dfrac{Y^2}{b'^4}}.$$

Now, if S, S'' be the foci of the given ellipse,

$$a'^2b'^2\left(\frac{X^2}{a'^4}+\frac{Y^2}{b'^4}\right) = OS \,.\, OS';$$

also $$\frac{X^2}{a'^2}+\frac{Y^2}{b'^2}=1,$$

whence $$\frac{X^2}{a'^2}=\frac{a'^2}{a'^2-b'^2}\left(1-\frac{OS\,.\,OS'}{a'^2}\right),$$

$$\frac{Y^2}{b'^2}=\frac{b'^2}{a'^2-b'^2}\left(\frac{OS\,.\,OS'}{b'^2}-1\right);$$

and $$\frac{X^2}{a'^2}+\frac{Y^2}{b'^2}=\frac{1}{a'^2-b'^2}\left\{\frac{a'^2}{a^2}-\frac{b'^2}{b^2}-OS\,.\,OS'\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\right\}$$

$$=\frac{\lambda+OS\,.\,OS'}{a^2b^2},$$

whence $$\frac{OF_2}{\frac{a'}{a}-\frac{b'}{b}}=\frac{OF_1}{\frac{a'}{a}+\frac{b'}{b}}=\tfrac{1}{2}F_1F_2\sqrt{\frac{\lambda+OS\,.\,OS'}{OS\,.\,OS'}},$$

or the vector equation is

$$r_1\,.\,OF_2+r_2\,.\,OF_1=r\,.\,F_1F_2\sqrt{\frac{\lambda+OS\,.\,OS'}{OS\,.\,OS'}}.$$

We have in our work at one point assumed that O is within the given ellipse, and, of course, the sign of r is doubtful; if O be without the ellipse, the vector equation will be

$$r_1\,.\,OF_2-r_2\,.\,OF_1\pm r\,.\,F_1F_2\sqrt{\frac{\lambda+OS\,.\,OS'}{OS\,.\,OS'}}=0,$$

the change of sign in the ambiguity taking place when r vanishes.

When O is without the ellipse, let 2α be the angle between the tangents drawn from O to the given ellipse; then

$$\cos 2\alpha=\frac{CO^2-a^2-b^2}{OS\,.\,OS'},\quad 2\sin^2\alpha=\frac{OS\,.\,OS'+a^2+b^2-CO^2}{OS\,.\,OS'};$$

also $$CO^2+OS\,.\,OS'=a^2-\lambda+b^2-\lambda,$$

whence $$\sin^2\alpha=\frac{\lambda+OS\,.\,OS'}{OS\,.\,OS'};$$

and, finally, the vector equation is

$$r_1\,.\,OF_2-r_2\,.\,OF_1\pm r\,.\,F_1F_2\sin\alpha=0.$$

When O is within the ellipse, α becomes impossible by $\sin\alpha$ becoming greater than 1, and the vector equation will then be

$$r_1\,.\,OF_2+r_2\,.\,OF_1=r\,.\,F_1F_2\sqrt{\frac{OS\,.\,OS'+a^2+b^2-OC^2}{2OS\,.\,OS'}}.$$

The work requires scarcely any alteration if the given curve be an hyperbola $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\right)$: we must then draw a confocal hyperbola through O, $\left(\frac{X^2}{a'^2}-\frac{Y^2}{b'^2}=1,\ \text{where } a'^2+b'^2=a^2+b^2\right)$, and draw normals

OF_1, OF_2 to the two hyperbolas

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\left(\frac{ab\mp a'b'}{a'b\mp ab'}\right)^2,$$

such that the middle point of F_1F_2 is the point

$$\left(\frac{a^2X}{a^2+b^2},\ \frac{b^2Y}{a^2+b^2}\right).$$

The points F_1, F_2, so determined, are the single foci of the pedal, and the vector equation is precisely the same in form as for the ellipse,

$$\left(\frac{a'}{a}-\frac{b'}{b}\right)r_1+\left(\frac{a'}{a}+\frac{b'}{b}\right)r_2=2r,$$

or $$r_1\,.\,OF_2+r_2\,.\,OF_1=r\,.\,F_1F_2\sqrt{\frac{OS\,.\,OS'+OC^2-a^2+b^2}{2OS\,.\,OS'}}$$

(for an internal point), and

$$r_1\,.\,OF_2-r_2\,.\,OF_1\pm r\,.\,F_1F_2\sin\alpha=0$$

(for an external point).

The results for a parabola might, of course, be deduced; but I have worked them out independently.* Of course, the form

$$r_1\,.\,OF_2-r_2\,.\,OF_1\pm r\,.\,F_1F_2\sin\alpha=0$$

remains true.

The following special cases of the results here investigated may be mentioned:—(1) When O lies on the transverse axis (between the foci for the ellipse, beyond them for the hyperbola), the real single foci of the pedal with respect to O are the feet of the normals drawn from O to the similar, &c. conic touching the directrices of the given conic, and the vector equation is $e\,(r_1+r_2)=2r$. (2) When O lies on a fixed conic confocal with the given conic and of the same species, the foci F_1, F_2 of the pedal lie on two fixed conics (one on each), which are similar, concentric, and similarly situated to the given conic; and OF_1, OF_2 are normals to these loci, the middle point of F_1F_2 lying also on a fixed conic. (3) When O lies on the given conic, there is only one single focus (F) of the pedal, and its coordinates are (in the case of the ellipse) $X\frac{a^2+b^2}{a^2-b^2}$, $Y\frac{b^2+a^2}{b^2-a^2}$; where X, Y are coordinates of O; OF being, as before, normal at F to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{a^2+b^2}{a^2-b^2}\right)^2,$$

which is the locus of F. The vector equation becomes, as might be foreseen, an identity in this case.

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* [See following note.]

Determination of the Foci of the Pedal of a given Parabola with respect to any proposed Point; and Investigation of different Forms of the Vector Equation of the Pedal. *By* Professor Wolstenholme.

[*Read January 12th*, 1882.]

The equation of the pedal of the parabola $y^2 = 4ax$ with respect to a point O, whose coordinates are X, Y, is

$$a(y-Y)^2+(x-X)(x^2+y^2-xX-yY) = 0,$$

or, with the origin at O,

$$ay^2+x(x^2+y^2+xX+yY) = 0.$$

If, then, $x+iy = p$ be a tangent, $ay^2+px(x-iy)+x^2X+xyY$ must be a perfect square, or $(p+iY)^2+4a(p+X) = 0$;

whence, restoring the value $x+iy$ of p, and equating real and impossible parts, $x^2-(y+Y)^2+4a(x+X) = 0$, $x(y+Y)+2ay = 0$;

or, removing the origin back to the vertex, the equations giving the foci of the pedal will be

$$(x-X)^2 = y^2-4ax, \quad (x-X)y+2a(y-Y) = 0 \ldots\ldots (\mathrm{A}, \mathrm{B}).$$

These equations give us two real foci F_1, F_2, and two impossible foci, anti-points to F_1, F_2; and equation (B) proves that OF_1, OF_2 are normals at F_1, F_2 to two parabolas equal to the original and having the same axis. If we write this equation in the form

$$\frac{x-X}{2a} = \frac{y-Y}{-y} = k,$$

we get, by (A), $$y^2-4ax = 4a^2k^2,$$

or the two parabolas to which OF_1, OF_2 are normals are

$$y^2 = 4a(x+ak_1^2), \quad y^2 = 4a(x+ak_2^2);$$

and, since $$x = X+2ak, \quad y = \frac{Y}{1+k},$$

$$\frac{Y^2}{(1+k)^2} = 4a(X+2ak+ak^2),$$

or $$Y^2 = 4b(X+b-a),$$

where $$b = a(1+k)^2.$$

This quadratic for b has always two real roots corresponding to the two parabolas which can be drawn through O confocal with the

original; but, for the real foci, we must take the positive root, or the parabola whose concavity is in the same sense as that of the given parabola. We have then

$$k_1 = -1+\sqrt{\frac{b}{a}}, \quad k_2 = -1-\sqrt{\frac{b}{a}},$$

and OF_1, OF_2 are normals at F_1, F_2 to the two parabolas

$$y^2 = 4a\,[x+(\sqrt{b}\mp\sqrt{a})^2].$$

Also, if (x_1y_1), (x_2y_2) be the coordinates of F_1, F_2,

$$y_1+y_2 = \frac{Y}{1+k_1}+\frac{Y}{1+k_2} = 0,$$

$$x_1+x_2 = 2\,(X+ak_1+ak_2) = 2\,(X-2a);$$

so that $(X-2a,\ 0)$ is the middle point of F_1F_2.

To determine the vector equation, take the point $(0,\ Y)$ on the pedal, and let r, r_1, r_2 be its distances from O, F_1, F_2.

Then $r = X$, $r_1^2 = x_1^2+(y_1-Y)^2 = (X+2ak_1)^2+\dfrac{k_1^2\,Y^2}{(1+k_1)^2}$

$$= (r+2ak_1)^2+4ak_1^2\,(r+2ak_1+ak_1^2),$$

or
$$r_1 = r+2ak_1\,(1+k_1),$$

and similarly
$$r_2 = r+2ak_2\,(1+k_2).$$

Hence the vector equation is

$$k_2r_1+k_1r_2 = (k_1+k_2)\,r,$$

or
$$\left(1+\sqrt{\frac{b}{a}}\right)r_1+\left(1-\sqrt{\frac{b}{a}}\right)r_2 = 2r.$$

Also $OF_1^2 = 4a^2k_1^2+\dfrac{k_1^2\,Y^2}{(1+k_1)^2} = 4a^2k_1^2+4ak_1^2\,(X+2ak_1+ak_1^2)$

$$= 4ak_1^2\,(X+b).$$

Similarly
$$OF_2^2 = 4ak_2^2\,(X+b);$$

and
$$(F_1F_2)^2 = 4a^2\,(k_1-k_2)^2+\frac{Y^2}{(1+k_1)^2}\,(k_1+k_2)^2$$

$$= 16a\,(X+2b-a),$$

whence
$$\frac{OF_2}{1+\sqrt{\dfrac{b}{a}}} = \frac{OF_1}{1-\sqrt{\dfrac{b}{a}}} = \frac{F_1F_2}{2\sqrt{\dfrac{X+2b-a}{X+b}}};$$

and the vector equation may be written

$$r_1\,.\,OF_2+r_2\,.\,OF_1 = r\,.\,F_1F_2\sqrt{\frac{X+b}{X+2b-a}}.$$

It has been assumed here that O lies within the parabola so that $b < a$. When O is without the parabola, the equation will be

$$r_1 . OF_2 - r_2 . OF_1 \pm r . F_1F_2 \sqrt{\frac{X+b}{X+2b-a}} = 0,$$

and, if 2α be the angle between the two tangents drawn from O to the given parabola, $\cos 2\alpha = \dfrac{X+a}{a-2b-X}$, $2\sin^2\alpha = \dfrac{2(X+b)}{X+2b-a}$, or the vector equation is

$$r_1 . OF_2 - r_2 . OF_1 \pm r . F_1F_2 \sin\alpha = 0.$$

When O lies on the parabola, $b = a$, $OF_1 = 0$, or F_1 coincides with O, which is then a triple focus (and cusp) of the pedal, $k_1 = 0$, $k_2 = -2$, and the coordinates of F_2, the only real single focus, will be $X-4a$, $-Y$; OF_2 being normal at F_2 to the parabola $y^2 = 4a(x+4a)$. When O lies on the axis, on the opposite side of the focus to the vertex, $b = 0$, OF_1, OF_2 are normals drawn from O to the parabola $y^2 = 4a(x+a)$, and the vector equation is $r_1 + r_2 = 2r$.

On the Calculation of Symmetric Functions.

By Mr. J. Hammond, M.A.

[*Read January 12th*, 1882.]

1. Consider the equation

$$(x-\mu)(x^n - p_1x^{n-1} + p_2x^{n-2} - p_3x^{n-3} + \ldots) = 0 \ldots\ldots\ldots\ldots (1),$$

suppose the roots of the equation of the n^{th} degree to be $\alpha, \beta, \gamma, \ldots$, and let $\Sigma\alpha^q\beta^q \ldots \gamma^r\delta^s \ldots\ldots = (p_1, p_2, p_3, \ldots\ldots)$;

the introduction of the new root μ, on the one hand, changes $p_1, p_2, p_3, \ldots$ into $p_1+\mu$, $p_2+\mu p_1$, $p_3+\mu p_2$, ..., and, on the other, introduces into the symmetric function new terms of the form

$$\mu^q\Sigma\alpha^q \ldots \gamma^r\delta^s \ldots\ldots .$$

Hence

$$\Sigma\alpha^q\beta^q \ldots \gamma^r\delta^s \ldots\ldots + \mu^q\Sigma\alpha^q \ldots \gamma^r\delta^s \ldots\ldots + \mu^r\Sigma\alpha^q\beta^q \ldots \delta^s \ldots\ldots + \mu^s\Sigma\alpha^q\beta^q \ldots \gamma^r \ldots\ldots$$
$$= \phi(p_1+\mu, p_2+\mu p_1, p_3+\mu p_2, \ldots\ldots) \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

If now the expanded value of ϕ, in (2), is written

$$\phi + \mu\phi_1 + \mu^2\phi_2 + \mu^3\phi_3 + \ldots\ldots,$$

equating coefficients, we have

$$\left.\begin{aligned} \phi_\kappa &= 0 \text{ unless } \kappa = q, r, s \\ \phi_q &= \Sigma a^q \dots \gamma^r \delta^s \dots\dots \\ \phi_r &= \Sigma a^q \beta^q \dots \delta^s \dots\dots \\ \dots &\quad \dots \quad \dots \end{aligned}\right\} \dots\dots\dots\dots\dots\dots\dots\dots (3).$$

Or, the symmetric function equivalent to ϕ_κ is obtained by simply cancelling one of the letters in the original symmetric function that occurs with the index κ. If we wish to find the value of Σa^n,

$$\phi_1, \phi_2, \dots \phi_{n-1} = 0,$$

and $\phi_n = 1$.

2. Several terms of a symmetric function can often be found by inspection from known formulæ.

Thus, let $\phi = \Sigma a^q \beta^r \gamma^s \dots\dots,$

and suppose $\Sigma_m a^q \beta^r \gamma^s \dots$ to denote the value of the symmetric function when p_{m+1} and all succeeding coefficients in (1) are equated to zero; then

$$\begin{aligned} \phi(p_1, p_2, p_3, \dots p_n) &= \varphi(p_1, p_2, p_3, \dots p_m, 0, 0, \dots) \\ &\quad + \text{terms containing } p_{m+1}, p_{m+2}, p_{m+3}, \dots p_n \\ &= \Sigma_m a^q \beta^r \gamma^s \dots\dots + \text{other terms} \dots\dots\dots\dots\dots (4). \end{aligned}$$

In the case $\Sigma a^5 \beta^4 \gamma^3 \delta^2 \epsilon$, we have

$$\Sigma_5 a^5 \beta^4 \gamma^3 \delta^2 \epsilon = p_5 \Sigma_5 a^4 \beta^3 \gamma^2 \delta,$$

and, referring to tables of the 10th degree, we see that

$$\begin{aligned} \Sigma a^5 \beta^4 \gamma^3 \delta^2 \epsilon = p_5 (&p_1 p_2 p_3 p_4 - 3p_3^2 p_4 - 3p_1^2 p_4^2 + 4p_2 p_4^2 - 3p_1 p_2^2 p_5 + 4p_1^2 p_3 p_5 \\ &+ 5p_2 p_3 p_5 - 5p_5^2) \\ &+ \text{terms containing } p_6, p_7, p_8, \dots p_{15}. \end{aligned}$$

If $\phi = \Sigma a^n$, it is easily seen that

$$\begin{aligned} \Sigma_2 a^n = p_1^n - np_1^{n-2} p_2 + \frac{n(n-3)}{2!} p_1^{n-4} p_2^2 - \frac{n(n-4)(n-5)}{3!} p_1^{n-6} p_2^3 \\ + \frac{n(n-5)(n-6)(n-7)}{4!} p_1^{n-8} p_2^4 - \&c. \dots\dots (5). \end{aligned}$$

In fact, if we consider the equation

$$x^2 - 2x \cos\theta + 1 = 0,$$

in which $p_1 = 2\cos\theta$ and $p_2 = 1$, it is obvious that $\Sigma_2 a^n = 2\cos n\theta$, and hence that the coefficients in (5) are those of the development of $2\cos n\theta$ in descending powers of $2\cos\theta$.

This gives $\frac{n+1}{2}$ or $\frac{n}{2}+1$ of the leading terms of Σa^n according as n is odd or even.

3. The use of ϕ_1 alone will sometimes enable us to calculate the value of a symmetric function, and in the case where Σ contains no letters with unity for their index, the determination will be independent of the values of symmetric functions of inferior degrees, since here $\phi_1 = 0$.

For example, if $\phi = \Sigma\alpha^2\beta^2\gamma^2\ldots\ldots$, the index of the last letter being 1 or 2 according as n is odd or even.

First, when $n = 2m$, assume

$$\phi = p_m^2 + A_1 p_{m-1} p_{m+1} + A_2 p_{m-2} p_{m+2} + A_3 p_{m-3} p_{m+3} + \ldots \\ \ldots + A_{m-1} p_1 p_{2m-1} + A_m p_{2m};$$

operating with $\dfrac{d}{dp_1} + p_1 \dfrac{d}{dp_2} + p_2 \dfrac{d}{dp_3} + \ldots\ldots,$

we have

$$\phi_1 = p_{m-1} p_m (2 + A_1) + p_{m-2} p_{m+1} (A_1 + A_2) + p_{m-3} p_{m+2} (A_2 + A_3) + \ldots \\ \ldots + p_{2m-1} (A_{m-1} + A_m),$$

where, since in this case $\phi_1 = 0$, it follows that

$$A_1 = -2,\ A_2 = 2,\ \ldots\ldots\ A_m = (-)^m 2,$$

i.e., $$\Sigma\alpha^2\beta^2\gamma^2\ldots = p_m^2 - 2p_{m-1} p_{m+1} + 2p_{m-2} p_{m+2} - \ldots + (-)^m 2p_{2m} \quad \ldots(6).$$

Secondly, if $n = 2m+1$, assume

$$\phi = p_m p_{m+1} + B_1 p_{m-1} p_{m+2} + B_2 p_{m-2} p_{m+3} + \ldots + B_{m-1} p_1 p_{2m} + B_m p_{2m+1};$$

as before,

$$\phi_1 = p_m^2 + p_{m-1} p_{m+1} (1 + B_1) + p_{m-2} p_{m+2} (B_1 + B_2) + \ldots + (B_{m-1} + B_m) p_{2m};$$

but here, since the index of the last letter in ϕ is unity, this value of ϕ_1 must be equivalent to (6). Hence

$$1 + B_1 = -2,\ B_1 + B_2 = 2,\ \ldots\ldots\ B_{m-1} + B_m = (-)^m 2,$$

giving $$\Sigma\alpha^2\beta^2\gamma^2\ldots = p_m p_{m+1} - 3p_{m-1} p_{m+2} + 5p_{m-2} p_{m+3} - \ldots \\ \ldots + (-)^m (2m+1) p_{2m+1} \ldots\ldots(7).$$

Again, if $\phi = \Sigma\alpha^2\beta\gamma\delta\ldots = u_n$, the destruction of the last letter changes u_n into u_{n-1}, so that, if

$$u_n = p_1 p_{n-1} + A_n p_n,$$

$$\left(\frac{d}{dp_1} + p_{n-2}\frac{d}{dp_{n-1}} + p_{n-1}\frac{d}{dp_n}\right) u_n = u_{n-1} = p_1 p_{n-2} + (1 + A_n) p_{n-1}.$$

From this we obtain $$1 + A_n = A_{n-1},$$

and finally $$\Sigma\alpha^2\beta\gamma\delta\ldots = p_1 p_{n-1} - np_n \quad \ldots\ldots\ldots\ldots\ldots\ldots(8).$$

4. But ϕ_1, which is of weight $n-1$, will not in many cases give enough relations among the coefficients of ϕ, which is of weight n, to

determine all of them. In such cases the missing relations can be supplied by using single terms of $\phi_2, \phi_3, \ldots$; the method of finding such terms is in every case the same as in the following example, where $n = 11$, and it is required to find the coefficient of $p_1 p_2 p_3^2$ in ϕ_2.

If we consider all the factors of $p_1 p_2 p_3^2$, not exceeding in this case the second degree, viz., $1, p_1, p_2, p_3, p_1 p_2, p_1 p_3, p_2 p_3, p_3^2$, and the corresponding terms of the operator, in this case

$$\frac{1}{2!}\left(\frac{d^2}{dp_1^2} + 2p_1\frac{d^2}{dp_1 dp_2} + \ldots\right);$$

we see that the term $p_1 p_2 p_3^2$ in ϕ_2 can only come from

$$\tfrac{1}{2}\frac{d^2}{dp_1^2}(p_1^3 p_2 p_3^2), \quad p_1\frac{d^2}{dp_1 dp_2}(p_1 p_2^2 p_3^2), \quad p_2\frac{d^2}{dp_1 dp_3}(p_1^2 p_3^3),$$

$$p_3\frac{d^2}{dp_1 dp_4}(p_1^2 p_2 p_3 p_4), \quad p_1 p_2\frac{d^2}{dp_2 dp_3}(p_3 p_3^3), \quad p_1 p_3\frac{d^2}{dp_2 dp_4}(p_2^2 p_3 p_4),$$

$$p_2 p_3\frac{d^2}{dp_3 dp_4}(p_1 p_3^2 p_4), \quad \tfrac{1}{2}p_3^2\frac{d^2}{dp_4^2}(p_1 p_2 p_4^2);$$

and hence, if

$$\phi = \ldots\ldots + Ap_1^3 p_2 p_3^2 + Bp_1 p_2^2 p_3^2 + Cp_1^2 p_3^3 + Dp_1^2 p_2 p_3 p_4 + Ep_2 p_3^3 + Fp_2^2 p_3 p_4 + Gp_1 p_3^2 p_4 + Hp_1 p_2 p_4^2 + \ldots\ldots,$$

$$\phi_2 = \ldots\ldots + (3A + 2B + 6C + 2D + 3E + 2F + 2G + H)\, p_1 p_2 p_3^2 + \ldots\ldots,$$

and if we have already found all but one of the quantities $A, B, \ldots H$, the remaining one is found from ϕ_2.

The remaining article consists of applications of this principle, chiefly to the case Σa^n.

5. Suppose

$$\phi = A_0 p_1^n + A_2 p_1^{n-2} p_2 + A_3 p_1^{n-3} p_3 + \ldots + A_{n-2} p_1^2 p_{n-2} + A_{n-1} p_1 p_{n-1} + A_n p_n + \text{other terms},$$

then, if $\kappa < n$ the last terms of $\phi_1, \phi_2, \ldots \phi_\kappa$ are respectively

$$(A_{n-1} + A_n)\, p_{n-1}, \quad (A_{n-2} + A_{n-1})\, p_{n-2}, \quad \ldots (A_{n-\kappa} + A_{n-\kappa+1})\, p_{n-\kappa};$$

for the factors of $p_{n-\kappa}$ are simply 1 and $p_{n-\kappa}$, so that the last term of ϕ_κ is obtained from

$$\frac{1}{\kappa!}\frac{d^\kappa}{dp_1^\kappa}(A_{n-\kappa} p_1^\kappa p_{n-\kappa}),$$

and

$$\frac{1}{(\kappa-1)!}p_{n-\kappa}\frac{d^\kappa}{dp_1^{\kappa-1}\, dp_{n-\kappa+1}}(A_{n-\kappa+1} p_1^{\kappa-1} p_{n-\kappa+1}),$$

and is therefore $(A_{n-\kappa} + A_{n-\kappa+1})\, p_{n-\kappa}$.

When $\phi = \Sigma\alpha^n$, since from (5), or otherwise, we have $A_2 = -n$, it follows that $A_3 = n$, $A_4 = -n$, ... $A_n = (-)^{n+1} n$.

But if $\phi = \Sigma\alpha^q\beta^r$, where $q > r$ and $q + r = n$, we have

$$\phi_r = \Sigma\alpha^q, \quad \phi_q = \Sigma\alpha^r, \quad \phi_\kappa = 0,$$

for other values of κ; thus

$$A_q + A_{q+1} = \text{coeff. } p_q \text{ in } \Sigma\alpha^q = (-)^{q+1} q,$$

and

$$A_r + A_{r+1} = (-)^{r+1} r.$$

Here, then,

$$A_2, A_3, \ldots A_r = 0,$$

$$A_{r+1} = (-)^{r+1} r, \quad A_{r+2} = (-)^{r+2} r, \quad \ldots\ldots A_q = (-)^q r,$$

$$A_{q+1} = (-)^{q+1} (q + r) = (-)^{q+1} n, \quad A_{q+2} = (-)^{q+2} n, \quad \ldots\ldots A_n = (-)^n n.$$

These values must, however, be modified in the case

$$q = r = \frac{n}{2},$$

when

$$A_{q+1} = (-)^{q+1} \frac{n}{2}, \quad \ldots\ldots A_n = (-)^n \frac{n}{2}.$$

Other symmetric functions may be similarly treated.

Again, suppose

$$\phi = \ldots\ldots + A_m p_1^{n-m} p_m + B_m p_1^{n-m-2} p_2 p_m + A_{m+1} p_1^{n-m-1} p_{m+1} + \ldots\ldots,$$

and consider the term $p_1^{n-m-1} p_m$ in ϕ_1; factors $1, p_1, p_m$.

This term is

$$\frac{d}{dp_1}(A_m p_1^{n-m} p_m) + p_1 \frac{d}{dp_2}(B_m p_1^{n-m-2} p_2 p_m) + p_m \frac{d}{dp_{m+1}}(A_{m+1} p_1^{n-m-1} p_{m+1})$$

$$= \{(n-m) A_m + B_m + A_{m+1}\} p_1^{n-m-1} p_m, \text{ if } m > 2.$$

When $m = 2$,

$$\phi = \ldots\ldots + A_2 p_1^{n-2} p_2 + B_2 p_1^{n-4} p_2^2 + A_3 p_1^{n-3} p_3 + \ldots\ldots,$$

and

$$\phi_1 = \ldots\ldots + \{(n-2) A_2 + 2B_2 + A_3\} p_1^{n-3} p_2 + \ldots\ldots .$$

Hence $A_2, A_3, \ldots A_n$ having been found from the last terms of $\phi_1, \phi_2, \ldots$; $B_2, B_3, \ldots$, can be found from ϕ_1 only.

If $\phi = \Sigma\alpha^n$, since

$$A_2 = -n, \quad \ldots A_m = (-)^{m+1} n,$$

it follows that

$$B_2 = \frac{n(n-3)}{2},$$

which agrees with (5), and

$$B_m = (-)^m n (n-m-1), \quad m > 2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

the last of the series being

$$B_{n-2} = (-)^n n = \text{coeff. } p_2 p_{n-2} \text{ in } \Sigma a^n, \quad n > 4.$$

If, as before, $m > 2$, the term $p_1^{n-m-2} p_m$ in ϕ_2, whose factors are 1, p_1, p_m, p_1^2, $p_1 p_m$, can only originate from the five terms

$$\tfrac{1}{2} \frac{d^2}{dp_1^2} (p_1^{n-m} p_m), \quad p_1 \frac{d^2}{dp_1 dp_2} (p_1^{n-m-2} p_2 p_m),$$

$$p_m \frac{d^2}{dp_1 dp_{m+1}} (p_1^{n-m-1} p_{m+1}), \quad \frac{p_1^2}{2} \frac{d^2}{dp_2^2} (p_1^{n-m-4} p_2^2 p_m),$$

and

$$p_1 p_m \frac{d^2}{dp_2 dp_{m+1}} (p_1^{n-m-3} p_2 p_{m+1});$$

so that with coefficients $A_m \ldots B_m \ldots$ as before, if the coefficient of $p_1^{n-m-4} p_2^2 p_m$ in ϕ be denoted by C_m, the term in ϕ_2 is

$$\left\{ \frac{(n-m)(n-m-1)}{2} A_m + (n-m-2) B_m + (n-m-1) A_{m+1} + C_m + B_{m+1} \right\} p_1^{n-m-2} p_m,$$

whence the coefficients C_m are determined.

In the case Σa^n, substituting for the A's and B's their values found above, we obtain

$$C_m = (-)^{m+1} \frac{n(n-m-2)(n-m-3)}{2}, \quad m > 2 \quad \ldots\ldots\ldots (9),$$

the last term of the series being

$$C_{n-4} = (-)^{n+1} n.$$

The calculation of C_2, like that of B_2, will only reproduce a term of (5). Coefficients of other terms of Σa^n may be found in the same way, without the slightest variation in the method.

Thursday, February 9th, 1882.

S. ROBERTS, Esq., F.R.S., President, in the Chair.

Mr. J. H. Tompson, F.C.S., F.G.S., Science Master in the Auckland College, New Zealand, was elected a Member, and Mrs. Bryant was admitted into the Society.

Mr. Tucker read a short abstract of a paper by Mr. H. M. Jeffery, F.R.S., "On certain Quartic Curves having a Tacnode at Infinity at which the Line at Infinity is the Multiple Tangent."

Mr. J. W. L. Glaisher, F.R.S., having taken the Chair *pro tem.*, the President communicated some results connected with Euler's Formula connecting the sum of the Divisors of a Number with the Pentagonal Numbers, and remarked that the formula really expressed the equality of the sum of the divisors to the sum of the m^{th} powers of the roots of a certain equation.

Mr. Hammond and Mr. Tucker also made brief communications.

The following presents were received :—

Carte-de-visite likeness from Mr. C. E. Bickmore.

"Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux," 2e Série, Tome iv., 3e Cahier; Paris, 1881.

"Atti della R. Accademia dei Lincei—Transunti," Vol. vi., Fasc. 3°, 4°, 5°.

"Educational Times," February, 1882.

"Proceedings of Royal Society," Vol. xxxiii., No. 217.

"Tidsskrift for Mathematik .... Fjerde Række, Femte Aargang, Første, Andet, Tredie, Fjerde, Femte, Sjette Hefte;" Kjøberhavn, 1881.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome xvi., 3me, 4me, 5me Livraisons; Harlem, 1881.

"Jahrbuch über die Fortschritte der Mathematik," elfter Band, Heft 3; Jahrgang 1879; Berlin, 1882.

Donations to the Bodleian Library during the year ending November 8, 1881; Oxford, 1881.

"The Scientific Proceedings of the Royal Dublin Society," Vol. ii., Pt. vii. (November, 1880), Vol. iii., Pt. i. (January, 1881), Pt. ii. (April, 1881), Pt. iii. (July, 1881), Pt. iv. (October, 1881).

"The Scientific Transactions of the Royal Dublin Society," Vol. i., Series ii., Pt. xiii., "On the Possibility of Originating Wave Disturbances in the Ether by means of Electric Forces," Pt. 2, by G. F. Fitzgerald (read May 19th, 1880). Pt. xiv., "Explorations in the Bone Cave of Ballynamintra, near Cappagh, County Waterford," by A. L. Adams, G. H. Kinahan, and R. J. Ussher (read November 15th, 1880).

The following three Mémoires (des xii. Bandes der "Abhandlungen der Mathematisch-Physischen Classe der Königl.-sachsischen Gesellschaft der Wissenschaften", :—

C. Neumann: "Uber die Peripolaren Coordinaten," November (Leipzig, 1880).

C. Neumann: "Die Vertheilung der Elektricität auf einer Kugelcalotte," November 1 (Leipzig, 1880).

W. G. Hankel: "Elektrische Untersuchungen, fünfzehnte Abhandlung über die aktino- und Piezo-elektrischen Eigenschaften des Bergkrystalles und ihre Beziehung zu den Thermo-elektrischen," No. vii. (Leipzig, 1881).

On certain Quartic Curves having a Tacnode at Infinity at which the line at infinity is the Multiple Tangent. *By* Henry M. Jeffery.

[*Read Feb. 9th*, 1882.]

1. The quartics, which will be here classified, are comprised in the equation $\kappa\alpha^4 = u_2$,

and may be considered in three divisions, according as the satellite-conic ($u_2 = 0$), which has double contact with the quartic of the third order in each case, is hyperbolic, elliptic, or parabolic. Varieties occur in these three main divisions, if the quadruple asymptote ($\alpha = 0$) pass through the centre of this conic, or is parallel to one of its asymptotes.

Since there is a tacnode at the extremity of the quadruple asymptote ($\alpha = 0$), at which the line at infinity is the multiple tangent, the class is at most the eighth, or the quartic may be reduced by critical values, and otherwise, to the sixth, fifth, and fourth classes.

2. The three main divisions are denoted with two parameters (κ, λ),

$$(1)\ldots\ldots\ \kappa\alpha^4 = \beta\gamma + \lambda,$$
$$(2)\ldots\ldots\ \kappa\alpha^4 = l\beta^2 + m\gamma^2 + \lambda,$$
$$(3)\ldots\ldots\ \kappa\alpha^4 = \beta^2 + n\gamma + \lambda.$$

It is proposed to examine every possible case which can arise for varying values of κ, λ.

With this view, since changes can only occur (1) for critical quartics, (2) when the folium-depressions in an oval shrink to a folium-point or point of undulation, two mutual relations are investigated between κ, λ,—(1) when the quartics in a group are critical, (2) when they have folium-points.

These mutual relations are (1) a quintic, quartic, or quadric in (κ, λ), and (2) the (λ) or (κ) axes of coordinates or parallels to them; these relations are here designated the first and second discriminating curves. Beside the tacnode, real or imaginary, there may be a crunode or an acnode; and all these quartics are bifolium, unifolium, or non-folium. The tacnode itself may be merged in a tacnode-cusp, a triple-point, or a stapete-point, thereby reducing the class of these quartics from the eighth to the fifth or fourth respectively.

3. To determine the tangential equivalent equation to a group of quartics with quadruple asymptotes, as above defined.

Let the quartic assume the general form

$$\kappa x^4 = ax + 2bxy + cy^2 + 2ex + 2dy + \lambda.$$

Combine it in the usual way with the line $(x\xi+y\eta=1)$ to make the binary quartic homogeneous.

Then, if (ξ, η) touch the quartic, $S^3=27T^2$, where S, T are the quartic and sextic invariants,

$$S=\tfrac{1}{12}\phi^2-\kappa(\lambda\eta^4+2d\eta^3+c\eta^2),$$

where ϕ denotes, for brevity, the ends of the asymptotes, $a\eta^2-2b\xi\eta+c\xi^2$,

$$T=-\tfrac{1}{216}\phi^3-\tfrac{1}{6}\kappa\phi\eta^2(\lambda\eta^2-d\eta-\tfrac{1}{2}c)$$
$$+\tfrac{1}{4}\kappa\eta^4(e\eta-d\xi)^2+\tfrac{1}{2}\kappa\eta^4[(bd-ce)\xi+(be-ad)\eta]-\tfrac{1}{4}\kappa\eta^4(ac-b^2).$$

(T is most readily calculated from its determinant-form.) The class is hence seen to be in general the eighth, since η^4 disappears as a factor from the equation to the class-curve

$$S^3=27T^2.$$

There is usually a tacnode at the end of the (y) axis, and these quartics, if they have no other singularities, belong to Dr. Salmon's Genus IV.

4. To determine the first discriminating curve, or the mutual relation which subsists between the parameters κ, λ, when there are critical quartics in this group.

Let the group of quartics be thus denoted

$$\kappa x^4=ax^2+2bxy+cy^2+2ex+2dy+\lambda.$$

For critical values, $\quad 2\kappa x^3=ax+by+e,$

$$0=bx+cy+d,$$

(the locus of nodes, viz., the axis of figure of the quartic, which also bisects the asymptotic triangle,)

$$0=2\lambda+3ex+3dy+ax^2+2bxy+cy^2.$$

The first discriminating curve is the quintic eliminant of

$$2\kappa cx^3=(ac-b^2)x+ce-bd,$$

$$0=2\lambda c-2d^2+3(ce-bd)x+(ac-b^2)x^2.$$

The asymptotes $(c\lambda=d^2)$, and $(\kappa=0)$, correspond to the values o, ∞ of x. The point $(\kappa=0,\ \lambda+ex+dy=0)$ corresponds to the centre of the conic $\quad x_1 : y_1 : 1 :: bd-ce : be-ad : ac-b^2.$

Cor. 1.—If $a=c=e=0$, or one of the satellite-asymptotes is also a quadruple asymptote,—this first curve becomes a line through the origin $(b^4\lambda=d^4\kappa)$. See infra, § 13.

Cor. 2.—If the satellite-conic be resolved into two lines, this first curve becomes the (λ) axis.

Cor. 3.—If the satellite-conic be a parabola, $a=0$, $b=0$, this first

discriminating curve is a quartic with folium- and stapete-points

$$16\kappa\,(\lambda c - d^2)^3 + 27c^3e^4 = 0.$$

Cor. 4.—If $b = c = 0$, or the quadruple asymptotes are diameters of the parabola, there are no critical quartics. See § 15.

Cor. 5.—If the quadruple asymptote ($a = 0$) pass through the centre of the conic, $e = 0 = d$, and the locus is a hyperbola (see Fig. 1, § 8),

$$\kappa\lambda\,\{4c^2\kappa\lambda + (ac - b^2)^2\} = 0.$$

5. The first discriminating curve may also be usually obtained as the discriminant of a binary quantic.

Taking an axis of the conic as the (x) axis, the transformed equation becomes $\quad -\kappa x^4 + ax^2 + cy^2 + 2ex + \lambda = 0 \equiv f(x).$

At a double point, $\quad \dfrac{df}{dx} = 0,$ and $y = 0.$

The discriminant is $\quad S^3 = 27T^2,$

where $\quad S = -\kappa\lambda + \tfrac{1}{12}a^2,$

$$T = -\tfrac{1}{6}\kappa\lambda a - \tfrac{1}{216}a^3 + \tfrac{1}{4}\kappa e^2.$$

When expanded, besides the factor $\kappa = 0$, for the (λ) axis, this equation is $\quad \kappa^2\lambda^3 + \tfrac{1}{2}\kappa\lambda^2 a^2 - \tfrac{9}{4}\kappa\lambda a e^2 + \tfrac{27}{16}\kappa e^4 + \tfrac{1}{16}\lambda a^4 - \tfrac{1}{16}a^3e^2 = 0.$

But this process may merely obtain points of contact.

6. To determine the second discriminating curve, or the mutual relation between the parameters, when quartics of this group have points of undulation.

At a folium-point, or point of undulation, two stationary points are consecutive. Hence $\quad \dfrac{d^2y}{dx^2} = 0, \quad \dfrac{d^3y}{dx^3} = 0.$

If these tests be applied to quartics of this group,

$$\kappa x^4 = ax^2 + cy^2 + 2ex + \lambda,$$

$$2\kappa x^3 = ax + e + cy\,\frac{dy}{dx},$$

$$6\kappa x^2 = a + c\left(\frac{dy}{dx}\right)^2,$$

$$\kappa x = 0.$$

(1) If $\kappa = 0$, the (λ) axis discriminates between the parameters for bifolium and non-folium quartics.

(2) If $x = 0$, $a\lambda = e^2$; and this parallel to the (κ) axis also discriminates the bifolium from the non-folium quartics, if a and c are of

opposite signs, or the quartics are hyperbolic. This appears also from the quartic equation

$$\kappa x^4 = (lx+my+n)\,(lx-my+n).$$

To each point in the line $(a\lambda = e^2)$, there corresponds a quartic with two folium-points.

In the hyperbolic division of quartics, if one of the satellite-asymptotes coincides with the line at infinity, each has a folium-point, as is evident from their equation

$$\kappa x^4 = 2cx+2dy+\lambda.$$

7. When folium-points occur on quartics of this group, they lie by pairs on quadruple asymptotes.

Let the quartic assume its most general form

$$\kappa x^4 = ax^2+2bxy+cy^2+2ex+2dy+\lambda.$$

If it be solved as a quadratic in y,

$$bx+cy+d = \pm\sqrt{\phi}\,(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A).$$

At a point of inflexion $\dfrac{d^2y}{dx^2} = 0.$

Hence $$2\phi\,\frac{d^2\phi}{dx^2} = \left(\frac{d\phi}{dx}\right)^2.$$

This sextic determines the abscissæ of the points of inflexion in each folium.

It may also be written to show, that this sextic has the same invariant as the binary quartic

$$\phi\,\frac{d^2\phi}{dx^2} = H\,(\phi),$$

where $H\,(\phi)$ denotes the Hessian, which contains the same square factor as its primitive, if it has one.

Its geometrical meaning is thus explained in this case. When the binary quartic $\phi\,(x) = 0$ has equal roots, the quartic oval shrinks to a point, and two equal values of the sextic unite at the same point.

Resuming the thesis, it appears from (A) that for every value of (y), there are two of (x) equi-distant from the polar of the tacnode

$$ax+by+e = 0.$$

Points of inflexion, and consequently, in their limiting form of union, folium-points lie on quadruple asymptotes.

To apply these processes, we will commence with simple cases.

8. To determine all the quartics with quadruple asymptotes, whose satellite-hyperbola has its asymptotes concurrent with one of the former.

Division I.

Let a group of these central quartics be thus denoted—

$$y^2 = \kappa x^4 + x^2 - \lambda \ldots\ldots\ldots\ldots (1).$$

The following sextic determines the abscissæ of its stationary points (§ 7) for one of the group,

$$(2\kappa x^2 + 1)\,(\kappa x^4 + x^2 - \lambda) = (6\kappa\lambda + 1)\,x^2 \ldots (2).$$

The first discriminating curve (§ 4) is

$$\kappa\lambda\,(4\kappa\lambda + 1)^2 = 0.$$

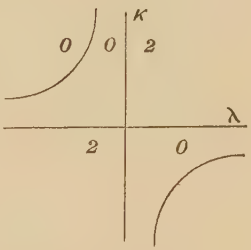

Fig. 1.

The corresponding quartics are resolved into two coaxal parabolæ

$$\kappa y^2 = (\kappa x^2 + \tfrac{1}{2})^2.$$

If the axes of κ, λ divide plane space, for critical points (κ, λ) in the second quadrant the quartics are resolved into coaxal parabolæ, and for critical values in the fourth quadrant the quartic ovals become mere points or acnodes.

If $\kappa = 0$, the quartics are resolved into hyperbolæ and the line at infinity, counted twice.

If $\lambda = 0$, the quartics are bi-flecnodal, and of the sixth class.

The axes of κ, λ also constitute the second discriminating curve (§ 6).

In the first quadrant let $\kappa = 1$, $\lambda = 1$.

The values of x^2 in (1) are $> \cdot 6$, and the only positive root in (2) is $1 \cdot 3$. Hence there are four inflexions, since the curve is symmetrical. The quartic is doubly campaniform, whose two branches meet in a tacnode at infinity, thus resembling a conchoid of Nicomedes.

In the third quadrant, let $\kappa = -1$, $\lambda = -1$.

The values are complementary to those obtained from the first quadrant.

In (1) $x^2 > 0 < 1 \cdot 62$, and in (2) x^2 has a value $1 \cdot 3$, the other two being negative and positive $> 1 \cdot 6$.

Hence the quartic is a bifolium-oval, like a double purse.

In the second quadrant, λ is negative, and two cases occur, as $\kappa > < -\frac{1}{4\lambda}$.

The values of x^2 in (2) are negative, and unlimited in (1).

Hence the quartics are non-folium, resembling the pair of parabolæ, which is their limiting and critical form : the companion-quartics do or do not intersect the (λ) axis, as $\kappa < > -\frac{1}{4\lambda}$.

In the fourth quadrant, κ is negative, and $< \frac{1}{4\lambda}$.

If $\kappa = \frac{1}{4\lambda}$, $y^2 + \lambda\left(1 - \frac{x^2}{2\lambda}\right)^2 = 0$, the oval shrinks to a point.

If $\kappa < \frac{1}{4\lambda}$, the real value of x^2 in (2) is beyond the limits of the oval, as defined by (1).

The numbers in the diagrams denote the number of folium-depressions.

9. To determine all the quartics with a quadruple asymptote, which passes through the centre of its satellite ellipse.

Division II.

Let a group of these central quartics be thus denoted— $y^2 = \kappa x^4 - x^2 - \lambda$.

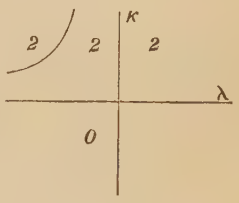

Fig. 2.

If $-x^2$ be substituted for $+x^2$ in § 8, the analysis is applicable, and the results seen to be complementary to those there obtained.

In the first quadrant, if $\kappa = \lambda = 1$, all the values of $x^2 > 1{\cdot}6$, and in the sextic (2) only one positive value exceeds this limit.

The quartic is bifolium, and doubly campaniform.

In the third quadrant, if $\kappa = \lambda = -1$, the values of x^2 in (1) are $> 0 < {\cdot}62$, whereas the positive value of (2) is 1·3. The quartic is a non-folium oval.

In the second quadrant, λ is negative, and two cases should be considered.

If $\kappa = -\frac{1}{4\lambda}$, the quartic is resolved into two coaxal parabolæ.

If $\kappa = 1$, $\lambda = -1$, the real value of the sextic (2) is ·2, and x^2 may have all values in (1). The quartic is bifolium and doubly campaniform.

If $\kappa = 1$, $\lambda = -{\cdot}1$, the values of x^2 in (1) are $< {\cdot}112$ and $> {\cdot}887$.

The only possible value of x^2 in (2) is 1·3. This quartic is also bifolium, but bipartite with a small oval about the origin.

In the fourth quadrant, no quartic is possible, so that the lower branch of the first discriminating hyperbola is inapplicable.

10. To determine all the quartics with a quadruple asymptote, whose satellite-hyperbola is unrestricted. (Division I.)

Let a group in this general case have this equation $y^2 = \kappa x^4 + x^2 + 2x - \lambda$ .....(1), the satellite-conic being an equilateral hyperbola.

The following sextic gives the abscissæ of the points of inflexion in each folium :—

$$(6\kappa x^2+1)(\kappa x^4+x^2+2x-\lambda) = (2\kappa x^3+x+1)^2 \text{.....(2).}$$

The first discriminating curve consists of a quintic (§ 4), and the (λ) axis,

$$\kappa(16\kappa^2\lambda^3+8\kappa\lambda^2+36\kappa\lambda+27\kappa+\lambda+1)=0.$$

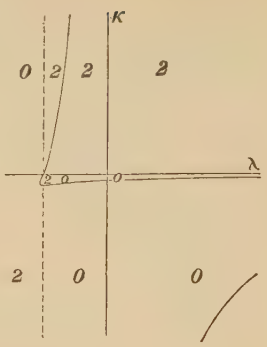

Fig. 3.

The second discriminating curve consists of two lines (§ 6),

$$\kappa = 0, \quad \lambda + 1 = 0.$$

If κ is positive, real branches touch the line at infinity in a tacnode ; if κ is negative, the tacnode is imaginary, and the quartics consist of one or two closed ovals.

Critical values occur in all quadrants but the first.

The description given in § 8 is here applicable, except for the looped portion in the third and fourth quadrants.

For points on the loop, the quartic is acnodal, and for values of (κ, λ) inside the loop, the quartic is bipartite, one of the ovals being bifolium, if it lie to the left of the discriminating line ($\lambda + 1 = 0$),* and in other cases non-folium.

11. To determine all the quartics with a quadruple asymptote, whose satellite-ellipse is unrestricted. (Division II.)

Let such a group be thus denoted—

$$y^2 = \kappa x^4 - x^2 - 2x - \lambda.$$

The satellite-conic is selected, without sacrificing generality, to be the circle of curvature at the points of double contact.

Now, the calculations necessary to draw the quartics corresponding to the several positions of (κ, λ) in the hyperbolic group of § 10, will suffice here also, if $-\kappa$, $-\lambda$ be written for κ, λ.

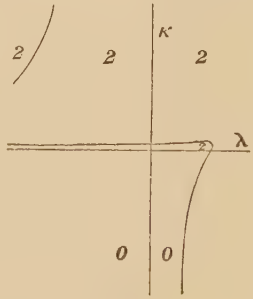

Fig. 4.

The first discriminating curve is identical with the former, when turned upside down.

The second discriminating line is the (λ) axis alone, since $y^2 + (x+1)^2$ denotes a pair of imaginary lines.

* The true position of the digit (2) in the wood-cut can hardly be discerned.

The description given in § 10 can be adopted, except for the loop in the first and second quadrants.

For values of (κ, λ) on the loop, the quartics are crunodal, and values within the loop, bipartite; the inner portion is a closed oval, and the outer doubly campaniform, whose branches unite and touch the line at infinity at the tacnode.

If (κ, λ) is on the first discriminating curve in the fourth quadrant, the quartic is reduced to an acnode.

The preceding quartics are of the eighth class, unless they are critical, when they are of the sixth class.

12. In the special case, when one satellite-asymptote coincides with the line at infinity, the tacnode at infinity becomes a stapete-point, and the quartics are of the fourth class. (Division I.) (Dr. Salmon's Genus IX.)

If the equation to the group be taken to be

$$\kappa x^4 = 2ex + 2dy + \lambda,$$

its tangential equivalent is (§ 3), after rejecting η^8, since two cusps and a node are merged in a stapete-point,

$$16\kappa\eta\,(\lambda\eta + 2d)^3 + 27\,(e\eta - d\xi)^4 = 0.$$

There is no critical quartic, and all have points of undulation (§ 6).

13. In the special case, when one satellite-asymptote is also a quadruple asymptote, the tacnode becomes either a tacnode-cusp or a triple point, and the quartics are of the fifth class. (Dr. Salmon's Genus VIII.) (Division I.)

Let the equation to the group be

$$\kappa x^4 = 2bxy + 2dy + \lambda.$$

Then, by § 3, the tangential equivalent is found from the invariants

$$S = \tfrac{1}{3} b^2\xi^2\eta^2 - \eta\kappa^3\,(\lambda\eta + 2d),$$

$$T = \tfrac{1}{27} b^3\xi^3\eta^3 + \tfrac{1}{3}\kappa b\xi\eta^4\,(\lambda\eta - d) + \tfrac{1}{4}\kappa\eta^4\,(d\xi + b)^2.$$

In simplifying the equivalent $(S^3 = 27T^2)$, η^7 is rejected, which represents a cusp at infinity on the line $(bx + d = 0)$, and the tacnode at infinity.

The line $(b^4\lambda = d^4\kappa)$ is the locus of points (κ, λ), which correspond to quartics, which are resolved into cubics and lines

$$\kappa\,(b^4x^4 - d^4) = 2b^4\,(bx + d)\,y.$$

The axes $(\kappa = 0, \lambda = 0)$ distinguish unifolium from non-folium quartics. The unifolium quartics have two concurrent cusps at the extremity of the (y) axis, whose dual form exhibits two points of inflexion on a common stationary tangent, the (x) axis. The non-folium quartics

have a triple point, in the same position at infinity, at which an ordinary tangent passes through a cusp. The former curves may be viewed as quartic tridents, with the folium-depression in each for a handle, the latter as Beta-shaped.

14. To determine all the quartics with quadruple asymptotes, whose satellite-parabola is unrestricted. (Division III.)

Such a group may be written

$$\kappa x^4 = cy^2 + 2ex + \lambda,$$

whose first discriminating curve is the quartic of fourth class and order

$$16\kappa\lambda^3 + 27e^4 = 0.$$

The (λ) axis discriminates the bifolium from the non-folium quartics; points (κ, λ) above the (λ) axis give the former curves.

By writing the equation to the critical curves in the form

$$16\lambda^3 y^2 + (2\lambda + 3ex)^2 \{(ex - 2\lambda)^2 + 2e^2x^2\} = 0,$$

it is seen that acnodes correspond to positive values of λ, and crunodes to negative values.

Non-singular quartics in this division are of the eighth class.

If $e=0$, the satellite is resolved into two parallel lines, or asymptotes whose points of contact are points of undulation ($cy^2 + \lambda = 0$).

The class of these quartics is also the eighth.

15. To determine all the quartics with quadruple asymptotes, which are also the diameters of their satellite parabola. (Division III.) (Dr. Salmon's Genus IX.)

Let the equation to this group be

$$\kappa x^4 = ax^2 + 2dy + \lambda.$$

Each quartic has a stapete-point at infinity, which reduces the class to the fourth.

The tangential equivalent is

$$S^3 = 27T^2,$$

where

$$\frac{S}{\eta^3} = \tfrac{1}{12}\eta - \kappa(\lambda\eta + d),$$

$$\frac{T}{\eta^4} = \tfrac{1}{216}\eta^2 - \tfrac{1}{6}\kappa\eta(\lambda\eta - d) + \tfrac{1}{4}\kappa d^2\xi^2 - \tfrac{1}{2}\kappa d\eta.$$

It is seen that η^8 is rejected as a factor, so that the class is the fourth.

There are no critical quartics in this group, but the hyperbola

$$4\kappa\lambda + a^2 = 0$$

determines the (x) axis to be a bitangent.

The (λ) axis discriminates between unifolium and non-folium quartics.

16. To show that there is a focus in the tacnode.

Let the group be denoted by the trilinear equation

$$\kappa a^4 \alpha^4 = bc\beta\gamma\,(2\Delta)^2 + \lambda\,(2\Delta)^4.$$

If the line $(a\alpha p + b\beta q + c\gamma r = 0)$ touches the quartic, $S^3 = 27T^2$, where

$$S = \tfrac{1}{12}\,(r-p)^2\,(p-q)^2 - \kappa\lambda\,(r-\phi)^4 + \kappa qr\,(r-q)^2,$$

$$T = -\tfrac{1}{2}\tfrac{1}{16}\,(r-p)^3\,(p-q)^3 - \tfrac{1}{6}\kappa\lambda\,(r-q)^4\,(p-q)(r-p)$$
$$+\kappa\,(r-q)^2\,\{\tfrac{1}{16}p^2\,(r-q)^2 - \tfrac{1}{12}\,(r-p)\,(p-q)\,qr\}.$$

The class is the eighth, as has been shown, $(r-q)^4$ being rejected for the tacnode. When this octavic $(S^3 = 27T^2)$, thus reduced, is expanded in a series of powers of $(r-q)^2$, the first term is

$$\tfrac{1}{16}\kappa\lambda\,(r-p)^4\,(p-q)^4 - \tfrac{1}{16}\kappa^2 q^2 r^2\,(r-p)^2\,(p-q)^2.$$

If we substitute for $(r-p)\,(p-q)$ its value derived from the fundamental relation

$$4\Delta^2 = a^2\,(p-q)\,(p-r) + b^2\,(q-r)\,(q-p) + c^2\,(r-p)\,(r-q),$$

the octavic takes the form, which proves the property in question

$$(r-q)\,u_7 = 4\Delta^2 v_4,$$

since $(r-q=0)$ denotes the extremity of BC, the side of reference. There is no trace of an analogue to the theorem in Spherics, that if a quartic have a quadruple cyclic arc, the quadrantal pole of that arc is a quadruple focus.

In the case where one satellite-asymptote of these quartics coincides with the line at infinity, the focus at the extremity of the (y) axis is thus shown.

Let the equation of § 12, $(\kappa x^4 = 2dy + \lambda)$,

be transformed to $$\kappa x^4 = 2fy.$$

Its tangential equivalent is $27f\xi^4 + 32\kappa\eta = 0$.

Let the constants be so chosen that, for convenience,

$$2\xi^4 + \eta = 0.$$

By Plücker's definition of a focus (x', y'), the quartic must be touched by the imaginary asymptote drawn through it,

$$x - x' + i\,(y - y') = 0.$$

Hence the tangential equivalent must be satisfied by the coordinates

$$\xi : \eta : 1 :: 1 : i : (x' + iy')^{-1}.$$

Therefore $$2 + i\,(x+iy)^3 = 0,$$

with a factor $(x+iy)^{-1}$, if the accents are dropped from x', y'. So also

$$2 - i\,(x-iy)^3 = 0.$$

The foci are found from the simultaneous equations

$$\left.\begin{aligned} 2-y\,(3x^2-y^2) &= 0 \\ x\,(x^2-3y^2) &= 0 \end{aligned}\right\}.$$

Three real foci have the coordinates

$$x=0, \quad y+\sqrt[3]{2}=0\,; \quad x^2=3y^2, \quad y=\tfrac{1}{2}\sqrt[3]{2}.$$

The situation of the fourth as of the first real focus is indicated by the form of the tangential equation

$$-2\,(\xi^2+\eta^2)^2+4\eta^2\,(\xi^2+\eta^2)=\eta\,(2\eta^3+1).$$

In like manner, the foci may be found in all the other cases; and one real focus, single or double, lies at the extremity of the (y) axis.

17. The general form of quartics, which have a tacnode at infinity, and quadruple asymptotes, is not here considered,

$$(x+c_1)\,(x+c_2)\,(x+c_3)\,(x+c_4)=u_2.$$

By quadruple asymptote is meant any line, which meets the curve in two coincident points of the tacnode, in whose direction the directions of the four possible asymptotes to a quartic are merged.

It appears from § 4, Cor. 5, that a change in the coordinates of the quartic group alters the degree of the discriminating curve.

Thursday, March 9th, 1882.

S. ROBERTS, Esq., F.R.S., President, in the Chair.

Mr. A. Buchheim, B.A., Scholar of New College, Oxford, was elected a member.

The following communications were made:—

"In how many independent ways can a Polygon of $2n$ sides be divided into Quadrilaterals by means of non-intersecting diagonals? with an extension to the general case of division into p-gons:" Mr. H. M. Taylor and Prof. Rowe (by whom the paper was communicated).

"Systems of Formulæ for the *sn*, *cn*, and *dn* of $u+v+w$:" Prof. W. Woolsey Johnson (communicated by Mr. Glaisher).

"Remarks on the preceding paper," and "Some Elliptic Function Formulæ:" Rev. M. M. U. Wilkinson.

Two Notes—(1) "Proof that

$$\int_0^\infty \frac{X^{m-1}dx}{X^{2p+1}+1}=\frac{\pi}{(2p+1)\sin\dfrac{m\pi}{2p+1}},$$

if $m<2p+1$, and both m and p positive integers;" (2.) "On the surface

$$\frac{a^2x^2}{r^2(a^2-r^2)}+\frac{b^2y^2}{r^2(b^2-r^2)}+\frac{c^2z^2}{r^2(c^2-r^2)}=0,$$

its Form, Conical Points, and singular Tangent Planes:" Mr. C. E. Bickmore.

"Note of Proofs of the Addition-Theorem for the Second Elliptic Integral, and of Fagnani's Theorem by Confocal Conics:" Mr. J. J. Walker.

The following presents were received:—

"Educational Times," March, 1882.

"Reprint of Mathematics from the Educational Times," Vol. xxxvi.: from the Publishers.

"Beiblätter zu den Annalen der Physik und Chemie," Band vi., Stück 2, No. 2.

"Bulletin de la Société Mathématique de Paris," Tome ix., No. 5.

"Monatsbericht," December, 1881.

"Atti della R. Accademia dei Lincei—Transunti," Vol. vi., Fasc. 6°; Roma.

"The Mathematical Magazine, a Journal of Elementary Mathematics," edited and published by Artemas Martin, M.A., Erie, Pa., Pt. 1; 1882.

"Sitzungsberichte der Physikalisch-medicinischen Societät zu Erlangen," 13 Heft; November 1880, bis August 1881.

"Jornal de Sciencias Mathematicas e Astronomicas," publicado pelo Dr. Fr. Gomes Teixeira, Vol. iii., Nos. 4, 5, 6; Coimbra.

"Boletin de la Institucion libre de Enseñanza," Año vi., Dec. 31, 1881.

Systems of Formulæ for the sn, cn, *and* dn *of* $u+v+w$. *By* William Woolsey Johnson, Professor of Mathematics in the Naval Academy, Annapolis, Maryland, U.S.

[*Read March 9th*, 1882.]

1. In the *Messenger of Mathematics*, for July, 1881, Mr. Glaisher has given formulæ for the sn, cn, and dn of $u+v+w$, in which the common denominator is a rational function of sn^2u, sn^2v, and sn^2w; and he has also recorded formulæ first given by Professor Cayley, in which the common denominator is not rational, but which reduce when $w=0$ to the usual forms of the sn, cn, and dn of $u+v$, in which the common denominator is rational in sn^2u and sn^2v. This communication contains the two systems of formulæ for $u+v+w$ of which those just mentioned are members.

2. If for sn, cn and dn we substitute $\frac{s}{n}$, $\frac{c}{n}$ and $\frac{d}{n}$, and use the suffixes 1 and 2 to refer to the arguments u and v, and the large letters

S, C, D, N to refer to the argument $u+v$, the complete system of formulæ for the functions of $u+v$ consists of four sets of values of the ratios $S : C : D : N$. Distinguishing these values by the suffixes s, c, n, and d, we may write

$$S_n = s_1 n_1 c_2 d_2 + s_2 n_2 c_1 d_1, \qquad S_s = n_1^2 s_2^2 - s_1^2 n_2^2,$$
$$C_n = c_1 c_2 n_1 n_2 - s_1 s_2 d_1 d_2, \qquad C_s = s_1 c_1 d_2 n_2 - s_2 c_2 d_1 n_1,$$
$$D_n = d_1 d_2 n_1 n_2 - k^2 s_1 s_2 c_1 c_2, \qquad D_s = s_1 d_1 c_2 n_2 - s_2 d_2 c_1 n_1,$$
$$N_n = n_1^2 n_2^2 - k^2 s_1^2 s_2^2, \qquad N_s = s_1 n_1 c_2 d_2 - s_2 n_2 c_1 d_1;$$
$$S_c = s_1 c_1 d_2 n_2 + s_2 c_2 d_1 n_1, \qquad S_d = s_1 d_1 c_2 n_2 + s_2 d_2 c_1 n_1,$$
$$C_c = n_1^2 n_2^2 - s_1^2 n_2^2 - n_1^2 s_2^2 + k^2 s_1^2 s_2^2, \qquad C_d = c_1 c_2 d_1 d_2 - k'^2 s_1 s_2 n_1 n_2,$$
$$D_c = c_1 c_2 d_1 d_2 + k'^2 s_1 s_2 n_1 n_2, \qquad D_d = n_1^2 n_2^2 - k^2 s_1^2 n_2^2 - k^2 n_1^2 s_2^2 + k^2 s_1^2 s_2^2,$$
$$N_c = c_1 c_2 n_1 n_2 + s_1 s_2 d_1 d_2, \qquad N_d = d_1 d_2 n_1 n_2 + k^2 s_1 s_2 c_1 c_2.$$

Putting herein each n equal to unity, we return to the usual notation in which s, c, and d stand for sn, cn, and dn respectively.

In each set one of the expressions is rational, and the suffix of the set refers to the letter which takes a rational form in that set.

3. The identities which result from a comparison of the various values of sn $(u+v)$, &c., are of the form

$$S_n N_s = S_s N_n, \quad S_c N_d = S_d N_c, \text{ \&c.};$$

viz., we may in any product interchange the suffixes.

4. It is noticeable that the sixteen expressions give also the values of the functions of $u-v$; for example,

$$\text{sn}\,(u-v) = \frac{N_s}{N_n} = \frac{S_s}{S_n} = \frac{C_s}{C_n} = \frac{D_s}{D_n},$$

the suffixes being determined by the required function, and the large letter being arbitrary.

The product of a function of $u+v$ by a function of $u-v$ is the ratio of two of the sixteen expressions; for example,

$$\text{sn}\,(u+v)\,\text{cn}\,(u-v) = \frac{S_d}{N_d}\,\frac{C_c}{C_n} = \frac{S_c C_d}{N_n C_d} = \frac{S_c}{N_n}.$$

In like manner, using cs for $\frac{\text{cn}}{\text{sn}}$, nd for $\frac{1}{\text{dn}}$, &c.,

$$\text{cs}\,(u+v)\,\text{nd}\,(u-v) = \frac{C_n}{S_d};$$

the large letters depending on the function of $u+v$, and the suffixes upon the function of $u-v$.

5. Using the suffix 3 to refer to the argument w, and the letters S, C, D, N to refer to the argument $u+v+w$, the process by which Mr. Glaisher's formulæ were derived may be exhibited as follows.

Employing the forms S_n, C_n, D_n and N_n in adding the arguments w and $u+v$, we have S, C, D, and N proportional to

$$s_3 n_3 CD + c_3 d_3 SN,$$
$$c_3 n_3 CN - s_3 d_3 SD,$$
$$d_3 n_3 DN - k^2 s_3 c_3 SC,$$

and

$$n_3 N^2 - k^2 s_3^2 S^2;$$

then, if we multiply each of these expressions by

$$n_3^2 N'^2 - k^2 s_3^2 S'^2,$$

where the accented letters refer to the argument $u-v$, it is found that the products are all symmetrical functions of u, v, and w.

Since the ratios only of these products are considered, it follows from 4, that we may in all of them for NN', SN', &c., substitute N_n, S_n, &c. Adopting the expressions so found as the values S, C, D, and N, we have, for example,

$$N = (n_3^2 N^2 - k^2 s_3^2 S^2)(n_3^2 N'^2 - k^2 s_3^2 S'^2)$$
$$= n_3^4 N_n^2 - k^2 n_3^2 s_3^2 (S_n^2 + N_s^2) + k^4 s_3^4 S_s^2,$$

which, putting each $n = 1$, is

$$(1 - k^2 s_1^2 s_2^2)^2 - 2k^2 s_3^2 (s_1^2 c_2^2 d_2^2 + s_2^2 c_1^2 d_1^2) + k^4 s_3^4 (s_1^2 - s_2^2)^2,$$

and this reduces to Mr. Glaisher's expression denoted by N_n in the table below.

6. In like manner, if we employ the forms S_s, C_s, D_s, and N_s in forming expressions proportional to S, C, D, and N, the expressions are rendered symmetrical by multiplying by

$$n_3^2 S'^2 - s_3^2 N'^2,$$

and the products may be denoted by S_s, C_s, D_s, and N_s.

Again, a third set of values denoted by S_c, C_c, D_c and N_c, and a fourth set S_d, C_d, D_d, N_d may be derived from the corresponding forms for $u+v$. The complete system is as follows, the first set being Mr. Glaisher's formulæ.

$$S_n = s_1 s_2 s_3 \{ -(1+k^2) + 2k^2 \Sigma s_1^2 - k^2 (1+k^2) \Sigma s_1^2 s_2^2 + 2k^4 s_1^2 s_2^2 s_3^2 \}$$
$$+ \Sigma s_1 c_2 c_3 d_2 d_3 (1 + k^2 s_2^2 s_3^2 - k^2 s_3^2 s_1^2 - k^2 s_1^2 s_2^2),$$

$$C_n = c_1 c_2 c_3 (1 - k^2 \Sigma s_1^2 s_2^2 + 2k^4 s_1^2 s_2^2 s_3^2)$$
$$+ \Sigma s_2 s_3 c_1 d_2 d_3 (-1 + k^2 s_2^2 s_3^2 - k^2 s_3^2 s_1^2 - k^2 s_1^2 s_2^2 + 2k^2 s_1^2),$$

$$D_n = d_1 d_2 d_3 (1 - k^2 \Sigma s_1^2 s_2^2 + 2k^2 s_1^2 s_2^2 s_3^2)$$
$$+ k^2 \Sigma s_2 s_3 c_2 c_3 d_1 (-1 + k^2 s_2^2 s_3^2 - k^2 s_3^2 s_1^2 - k^2 s_1^2 s_2^2 + 2 s_1^2),$$

$$N_n = 1 - 2k^2 \Sigma s_1^2 s_2^2 + 4k^2 (1+k^2) s_1^2 s_2^2 s_3^2 - 2k^4 \Sigma s_1^4 s_2^2 s_3^2 + k^4 \Sigma s_1^4 s_2^4,$$

$$S_s = \Sigma s_1^4 - 2\Sigma s_1^2 s_2^2 + 4(1+k^2)\, s_1^2 s_2^2 s_3^2 - 2k^2\Sigma s_1^4 s_2^2 s_3^2 + k^4 s_1^4 s_2^4 s_3^4,$$

$$C_s = s_1 s_2 s_3 c_1 c_2 c_3 (2 - k^2\Sigma s_1^2 + k^4 s_1^2 s_2^2 s_3^2) + \Sigma s_1 c_1 d_2 d_3 (s_1^2 - s_2^2 - s_3^2 + 2 s_2^2 s_3^2 - k^2 s_1^2 s_2^2 s_3^2),$$

$$D_s = s_1 s_2 s_3 d_1 d_2 d_3 (2 - \Sigma s_1^2 + k^2 s_1^2 s_2^2 s_3^2) + \Sigma s_1 d_1 c_2 c_3 (s_1^2 - s_2^2 - s_3^2 + 2k^2 s_2^2 s_3^2 - k^2 s_1^2 s_2^2 s_3^2),$$

$$N_s = s_1 s_2 s_3 \{2 - (1+k^2)\Sigma s_1^2 + 2k^2\Sigma s_1^2 s_2^2 - k^2(1+k^2) s_1^2 s_2^2 s_3^2\} + \Sigma s_1 c_2 c_3 d_2 d_3 (s_1^2 - s_2^2 - s_3^2 + k^2 s_1^2 s_2^2 s_3^2).$$

$$S_c = s_1 s_2 s_3 c_1 c_2 c_3 (2 - k^2 - k^2\Sigma s_1^2 + k^4\Sigma s_1^2 s_2^2 - k^4 s_1^2 s_2^2 s_3^2) + \Sigma s_1 c_1 d_2 d_3 \{1 - s_1^2 - s_2^2 - s_3^2 + (2-k^2) s_2^2 s_3^2 + k^2 s_3^2 s_1^2 + k^2 s_1^2 s_2^2 - k^2 s_1^2 s_2^2 s_3^2\},$$

$$C_c = 1 - 2\Sigma s_1^2 + 2(1+k^2)\Sigma s_1^2 s_2^2 + \Sigma s_1^4 - 2k^2\Sigma s_1^4 s_2^2 - 4(1+k^4) s_1^2 s_2^2 s_3^2 + 2k^2(1+k^2)\Sigma s_1^2 s_2^2 s_3^2 + k^4\Sigma s_1^4 s_2^2 - 2k^4\Sigma s_1^4 s_2^4 s_3^2 + k^4 s_1^4 s_2^4 s_3^4,$$

$$D_c = c_1 c_2 c_3 d_1 d_2 d_3 (1 - \Sigma s_1^2 + k^2\Sigma s_1^2 s_2^2 - k^2 s_1^2 s_2^2 s_3^2) + (1-k^2)\Sigma s_2 s_3 c_1 d_1 (1 + s_1^2 - s_2^2 - s_3^2 + k^2 s_2^2 s_3^2 - k^2 s_3^2 s_1^2 - k^2 s_1^2 s_2^2 + k^2 s_1^2 s_2^2 s_3^2),$$

$$N^c = c_1 c_2 c_3 \{1 - \Sigma s_1^2 - k^2\Sigma s_1^2 s_2^2 + k^2(1-2k^2) s_1^2 s_2^2 s_3^2\} + \Sigma s_2 s_3 c_1 d_2 d_3 \{1 + (1-2k^2) s_1^2 - s_2^2 - s_3^2 + k^2 s_2^2 s_3^2 + k^2 s_3^2 s_1^2 + k^2 s_1^2 s_2^2 - k^2 s_1^2 s_2^2 s_3^2\}.$$

$$S_d = s_1 s_2 s_3 d_1 d_2 d_3 (2k^2 - 1 - k^2\Sigma s_1^2 + k^2\Sigma s_1^2 s_2^2 - k^4 s_1^2 s_2^2 s_3^2) + \Sigma s_1 d_1 c_2 c_3 \{1 - k^2 s_1^2 - k^2 s_2^2 - k^2 s_3^2 - k^2(1-2k^2) s_2^2 s_3^2 + k^2 s_3^2 s_1^2 + k^2 s_1^2 s_2^2 - k^4 s_1^2 s_2^2 s_3^2\},$$

$$C_d = c_1 c_2 c_3 d_1 d_2 d_3 (1 - k^2\Sigma s_1^2 + k^2\Sigma s_1^2 s_2^2 - k^4 s_1^2 s_2^2 s_3^2) - (1-k^2)\Sigma s_2 s_3 c_1 d_1 (1 + k^2 s_1^2 - k^2 s_2^2 - k^2 s_3^2 + k^2 s_2^2 s_3^2 - k^2 s_3^2 s_1^2 - k^2 s_1^2 s_2^2 + k^4 s_1^2 s_2^2 s_3^2),$$

$$D_d = 1 - 2k^2\Sigma s_1^2 + 2k^2(1+k^2)\Sigma s_1^2 s_2^2 + k^4\Sigma s_1^4 - 2k^4\Sigma s_1^4 s_2^2 - 4k^2(1+k^4) s_1^2 s_2^2 s_3^2 + 2k^4(1+k^2)\Sigma s_1^4 s_2^2 s_3^2 + k^4\Sigma s_1^4 s_2^4 - 2k^6\Sigma s_1^4 s_2^4 s_3^2 + k^8 s_1^4 s_2^4 s_3^4,$$

$$N_d = d_1 d_2 d_3 \{1 - k^2\Sigma s_1^2 + k^2\Sigma s_1^2 s_2^2 + k^2(1-2k^2) s_1^2 s_2^2 s_3^2 + k^2\Sigma s_2 s_3 c_2 c_3 d_1 \{1 - (2-k^2) s_1^2 - k^2 s_2^2 - k^2 s_3^2 + k^2 s_2^2 s_3^2 + k^2 s_3^2 s_1^2 + k^2 s_1^2 s_2^2 - k^2 s_1^2 s_2^2 s_3^2\}.$$

7. Some of the above expressions were calculated in the manner explained in 5, but in most cases it was more convenient to derive them

by transformation from the first set. The second set may be derived by adding iK' to the argument u, which converts

$$s_1, \quad c_1, \quad d_1, \quad n_1,$$

respectively into

$$\frac{n_1}{k}, \quad -i\frac{d_1}{k}, \quad -ic_1, \quad s_1,$$

and produces a similar change in S, C, D, and N: or they may be found by adding iK' to each of the arguments, thereby adding $3iK'$ to $u+v+w$, and converting S, C, D, N into $\frac{N}{k}$, $\frac{iD}{k}$, iC, and S respectively.

In like manner, the third set may be derived by adding $K+iK'$ to one or all of the arguments, and the fourth set by adding K to one or all of the arguments.

The third set may also be obtained by transforming the first set to modulus k', and the fourth set by transforming the third set to the modulus $\frac{1}{k}$.

8. If $w=0$, the expressions reduce to the corresponding expressions for $u+v$. Putting $u=v=w$, they were found to reduce to the known expressions for $3u$, multiplied in the case of the second set by s^3, in the case of the third set by c^3, and in the case of the fourth set by d^3.

9. Professor Cayley's formulæ consist of the expressions denoted by S_n, C_n, D_n, N_n, in the system given below. In the case of these formulæ, the addition of iK', K, or $K+iK'$ to one of the arguments does not produce a symmetrical result, while the addition of the same quantities to each of the arguments merely reproduces the formulæ. But, by transforming to the modulus k', we have the expressions S_c, C_c, D_c, and N_c; and by transforming the latter to the modulus $\frac{1}{k}$, we have the expressions S_d, C_d, D_d, and N_d. The system, which appears to be complete without containing expressions of the form S_s, &c., is as follows:—

$$\begin{aligned}
S_n &= -s_1s_2s_3(1+k^2-k^2\Sigma s_1^2+k^4s_1^2s_2^2s_3^2) &&+\Sigma s_1c_2c_3d_2d_3,\\
C_n &= c_1c_2c_3(1-k^4s_1^2s_2^2s_3^2) &&-d_1d_2d_3\Sigma s_2s_3c_1d_1,\\
D_n &= d_1d_2d_3(1-k^2s_1^2s_2^2s_3^2) &&-k^2c_1c_2c_3\Sigma s_2s_3c_1d_1,\\
N_n &= 1-k^2\Sigma s_1^2s_2+k^2(1+k^2)s_1^2s_2^2s_3^2 &&-k^2s_1s_2s_3\Sigma s_1c_2c_3d_2d_3.
\end{aligned}$$

$$\begin{aligned}
S_c &= s_1s_2s_3(2-k^2-\Sigma s_1^2+k^2\Sigma s_1^2s_2^2-k^4s_1^2s_2^2s_3^2) &&+c_1c_2c_3\Sigma s_1c_1d_2d_3,\\
C_c &= c_1c_2c_3(1-\Sigma s_1^2+k^2\Sigma s_1^2s_2^2-k^4s_1^2s_2^2s_3^2) &&-(1-k^2)\,s_1s_2s_3\Sigma s_1c_1d_2d_3,\\
D_c &= d_1d_2d_3(1-\Sigma s_1^2+\Sigma s_1^2s_2^2-k^2s_1^2s_2^2s_3^2) &&+(1-k^2)\quad\Sigma s_2s_3c_2c_3d_1,\\
N_c &= 1-\Sigma s_1^2+\Sigma s_1^2s_2^2-k^2(2-k^2)s_1^2s_2^2s_3^2 &&+d_1d_2d_3\Sigma s_2s_3c_2c_3d_1.
\end{aligned}$$

$$S_d = s_1 s_2 s_3 (2k^2 - 1 - k^4 \Sigma s_1^2 + k^4 \Sigma s_1^2 s_2^2 - k^4 s_1^2 s_2^2 s_3^2) + d_1 d_2 d_3 \Sigma s_1 c_2 c_3 d_1,$$

$$C_d = c_1 c_2 c_3 (1 - k^2 \Sigma s_1^2 + k^4 \Sigma s_1^2 s_2^2 - k^4 s_1^2 s_2^2 s_3^2) - (1 - k^2) \Sigma s_2 s_3 c_1 d_2 d_3,$$

$$D_d = d_1 d_2 d_3 (1 - k^2 \Sigma s_1^2 + k^2 \Sigma s_1 s_2^2 - k^2 s_1^2 s_2^2 s_3^2) + k^2 (1 - k^2) s_1 s_2 s_3 \Sigma s_1 c_2 c_3 d_1,$$

$$N_d = 1 - k^2 \Sigma s_1^2 + k^4 \Sigma s_1^2 s_2^2 + k^2 (1 - 2k^2) s_1^2 s_2^2 s_3^2 + k^2 c_1 c_2 c_3 \Sigma s_2 s_3 c_1 d_2 d_3.$$

10. If we put $w = 0$, the second set reduce to the corresponding expressions for $u+v$ multiplied by $c_1 c_2$; and the third set reduce to the same quantities multiplied by $d_1 d_2$. When $u=v=w$, the formulæ reduce directly to the known expressions for $3u$.

It is noteworthy that the expressions for the functions of $u+v+w$, recently communicated to the Society by Mr. M. M. U. Wilkinson,* reduce, when $w=0$, to the forms S_s, C_s, D_s, and N_s; and that, when rendered homogeneous by the introduction of the letter n, they are symmetrical with respect to the four letters s, c, d, and n, and therefore constitute of themselves a complete system.

Note on a Geometrical Theorem. *By* H. M. Taylor and R. C. Rowe.

[*Read March 9th*, 1882.]

1. This note contains the solution of a generalized form of a problem interesting, if only from the names of the mathematicians whose attention it attracted half a century ago. The problem is,—"To find the number of independent ways in which a polygon of m sides can be divided into triangles by means of non-intersecting diagonals."

The solution seems to have been first discovered by Euler, who gives, but without proof, the number required for polygons up to the enneagon. The proof is supplied by J. A. de Segner, in a paper in the Petersburg Transactions [Nov. Comm., t. vii., pp. 203–210].

The question was next taken up in a series of four papers in Liouville's Journal for 1838–9. Proposed by Terquem, it was answered first by Lamé in a letter to Liouville (t. iii., pp. 505–7), and further discussed by O. Rodrigues (t. iii., pp. 547–8), M. J. Binet (t. iv., pp. 79–90), and E. Catalan (t. iv., pp. 91–94).

2. The result is most conveniently expressed in terms, not of the number (m) of angles of the polygon, but of the number (n) of triangles into which it can be divided. This number will be called the 'order' of the polygon; and it is clear that $m = n+2$. We write P_n for the required number of partitions in the case of a polygon of the n^{th} order.

* [See infra, pp. 106—109, and p. 70.]

3. Lamé, in the paper referred to above, arrives by geometrical reasoning at the two functional equations

$$(\alpha) \ldots\ldots P_{n+1} = P_n + P_1 P_{n-1} + P_2 P_{n-2} + \ldots + P_{n-1} P_1 + P_n,$$

$$(\beta) \ldots\ldots P_n = \frac{n+2}{2(n-1)} (P_1 P_{n-1} + P_2 P_{n-2} + \ldots + P_{n-1} P_1);$$

whence, immediately, it follows that

$$(\gamma) \ldots\ldots P_{n+1} = \frac{4n+2}{n+2} P_n.$$

Rodrigues gives an elegant direct demonstration of (γ). Binet's paper contains a solution by the method of generating functions with various 'reflections,' while Catalan, using a somewhat longer process, adds to the above equations a fourth,

$$(\delta) \ldots\ldots P_{n-1} - n P_{n-2} + \frac{n-1 \,.\, n-2}{1 \,.\, 2} P_{n-3} - \ldots = 0.$$

From (γ), together with the obvious relation $P_1 = 1$, we obtain at once

$$P_n = \frac{2n!}{n!\, n+1!}.$$

4. It does not seem to have occurred to any of these writers that the problem admits of extension to the case of division into polygons other than triangles, and none of their methods is alone sufficient for this purpose, though the solution for the general case is as easy as for the simple case of the triangle.

5. We notice first that there is a necessary relation between the number of sides of the polygon to be resolved and that of the polygons into which it is to be resolved. If the latter number be p, we find the number (m) of sides of the polygon which can be divided by diagonals into n p-gons (*i.e.*, the polygon which is "of the n^{th} order for division into p-gons") thus. There are clearly $n-1$ diagonals, and each of them is a side of two component polygons. All the sides of the m-gon, together with $n-1$ diagonals each counted twice, are therefore all the sides of n p-gons, so that

$$m + 2(n-1) = np,$$

or

$$m = (p-2)n + 2.$$

6. To find the number, P_n, of ways in which the m-gon, whose vertices we will call $A_1, A_2 \ldots A_m$, can be divided into n p-gons, we fix our attention on one side, for instance A_1A_2. This must be a side of one of the component polygons: suppose that, in a partition chosen at random, the other vertices of the p-gon which has A_1A_2 for a side are A_a, A_b, &c., the order being $A_1\, A_a\, A_b \ldots A_2$. It is obviously a necessary

condition that the polygon which has the angular points from A_1 to A_a inclusive for its vertices should be capable of division into p-gons; and let it be of the order a. Similarly, the polygon with the angular points from A_a to A_b for its vertices must be divisible into p-gons,—let its order be b, and so on. Then the first of these polygons is divisible into a p-gons in P_a ways; the second into b p-gons in P_b ways, and so on. It follows that the number of independent ways in which the m-gon can be divided into p-gons, *with the condition that the p-gon which has A_1A_2 for a side is* $A_1A_aA_b \dots A_2$, is $P_a P_b P_c \dots$; for the central polygon $A_1A_aA_b \dots A_2$ serves as a neutral zone to prevent mutual interference among the partitions of the polygons $A_1 \dots A_a$, $A_a \dots A_b$, &c.

Both the number and the magnitude of the quantities a, b, c, ... are subject to a condition; for, since there is one of these quantities corresponding to a polygon cut off by each side of the central polygon save A_1A_2, their number is $p-1$; also, the total number of resulting p-gons is obtained by adding unity for the central polygon to a, b, &c., for the polygons $A_1 \dots A_a$, $A_a \dots A_b$, &c.; we have, therefore, the further condition

$$a+b+c+\dots = n-1.$$

It is to be observed that, if any of the vertices A_1, A_a, A_b ... are consecutive, there will be a diminution in the number of the quantities a, b, c, &c.; for instance, if A_a were consecutive to A_1, we should have $a=0$. We can, however, allow for this by the introduction of a symbol P_0 equal in fact to unity. We have, then, the final result,

$$P_n = \Sigma P_a P_b P_c \dots, \quad \text{where} \quad a+b+c+\dots = n-1,$$

the number of the quantities a, b, c ... being $p-1$, and zero being admissible as a value for any of them.

7. This result may also be obtained by a method which will assign a meaning to the symbol

$$\Sigma P_a P_b P_c \dots \ (a+b+c+\dots = n-1),$$

where the number of factors in each term is less than $p-1$.

For the sake of clearness we will consider the case of division into pentagons, so that $p=5$, although the method is quite general. Let the polygon to be divided be $ABCD \dots XYZA$ and be of the order n (so that it has $3n+2$ sides).

Let S_n be the number of ways of division, subject to the condition that no diagonal shall pass through either of three assigned consecutive angular points, say A, Z, Y. Then, since B must be joined to X to form the pentagon $ABXYZ$, while the polygon cut off by this diagonal (which is of the order $n-1$) can then be divided in all possible independent ways, we have

$$S_n = P_{n-1}.$$

Next, let R_n be the number of ways of division subject to the condition that no diagonal shall pass through either of two assigned consecutive angular points, say Z, Y. Then this number = the number of partitions in which, as a further condition, no diagonal passes through $A+$ the number in which A is joined to no vertex further round the polygon (in the direction $ABC\ldots$) than $E+$ the number in which it is joined to no vertex further round than $H+\ldots$

$$= S_n + P_1 S_{n-1} + P_2 S_{n-2} + \ldots + P_{n-1} S_1$$
$$= P_0 P_{n-1} + P_1 P_{n-2} + P_2 P_{n-3} + \ldots + P_{n-1} P_0$$
$$= \Sigma P_a P_b \ (a+b=n-1).$$

Again, let Q_n be the number of ways of division subject to the condition that no diagonal passes through an assigned angular point, say Y. Then this number = the number of partitions in which no diagonal passes through $Z+$ the number in which Z is joined to no vertex further round than $D+$ the number in which it is joined to no vertex further round than $G+\ldots$

$$= R_n + P_1 R_{n-1} + \ldots + P_{n-1} R_1$$
$$= \Sigma P_a P_b P_c \ (a+b+c=n-1).$$

And, in the same way, we shall obtain

$$P_n = Q_n + P_1 Q_{n-1} + \ldots + P_{n-1} Q_1$$
$$= \Sigma P_a P_b P_c P_d \ (a+b+c+d=n-1).$$

Hence, in general, if P_n be the number of ways of dividing a polygon of order n into p-gons, the expression

$$\Sigma P_a P_b P_c \ldots (a+b+c+\ldots = n-1),$$

there being r factors in each term, is the number of ways of division under the condition that no diagonal is to pass through either of $p-r-1$ assigned consecutive angular points.

8. It only remains to solve the functional equation at the end of §6; which amounts to the statement that, if

$$f(x) = 1 + P_1 x + P_2 x^2 + \ldots \text{ ad. inf.},$$

then f satisfies the equation

$$f^{p-1} = \frac{f-1}{x}$$

or

$$f = 1 + x f^{p-1};$$

whence, by Lagrange's Theorem,

$$f = 1 + x\left[f^{p-1}\right] + \frac{x^2}{1.2}\left[\frac{d}{df} f^{2(p-1)}\right] + \&c.,$$

where, in the square brackets, f is to be put equal to unity after differentiation.

Hence
$$P_n = \frac{1}{n!}\left[\frac{d^{n-1}}{df^{n-1}} f^{n(p-1)}\right]$$
$$= (np-n)(np-n-1)\ldots(np-2n+2) \div n!$$
$$= \frac{\{(p-1)n\}!}{n!\,\{(p-2)n+1\}!},$$

which is therefore the number of independent ways in which a polygon of the order n (*i.e.*, a polygon of $(p-2)n+2$ sides) can be divided into p-gons.

Some Elliptic Function Formulæ. By the Rev. M. M. U. Wilkinson.

[*Read March 9th*, 1882.]

Since
$$k^2 \operatorname{sn}\alpha \operatorname{sn}\beta \operatorname{sn}(\alpha-\beta) = Z(\alpha-\beta) - Z\alpha + Z\beta,$$
$$k^2 \operatorname{sn}\beta \operatorname{sn}\gamma \operatorname{sn}(\beta-\gamma) = Z(\beta-\gamma) - Z\beta + Z\gamma,$$
$$k^2 \operatorname{sn}\gamma \operatorname{sn}\alpha \operatorname{sn}(\gamma-\alpha) = Z(\gamma-\alpha) - Z\gamma + Z\alpha,$$
$$k^2 \operatorname{sn}(\alpha-\beta) \operatorname{sn}(\beta-\gamma) \operatorname{sn}(\gamma-\alpha) = -Z(\alpha-\beta) - Z(\beta-\gamma) - Z(\gamma-\alpha);$$

we have

$$\operatorname{sn}\alpha \operatorname{sn}\beta \operatorname{sn}(\alpha-\beta) + \operatorname{sn}\beta \operatorname{sn}\gamma \operatorname{sn}(\beta-\gamma) + \operatorname{sn}\gamma \operatorname{sn}\alpha \operatorname{sn}(\gamma-\alpha)$$
$$= -\operatorname{sn}(\alpha-\beta) \operatorname{sn}(\beta-\gamma) \operatorname{sn}(\gamma-\alpha) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

a known formula. Substituting $\alpha+iK'$ for α, &c., we have

$$\operatorname{sn}\alpha \operatorname{sn}(\beta-\gamma) + \operatorname{sn}\beta \operatorname{sn}(\gamma-\alpha) + \operatorname{sn}\gamma \operatorname{sn}(\alpha-\beta)$$
$$= -k^2 \operatorname{sn}\alpha \operatorname{sn}\beta \operatorname{sn}\gamma \operatorname{sn}(\beta-\gamma) \operatorname{sn}(\gamma-\alpha) \operatorname{sn}(\alpha-\beta) \ldots (2),$$

since
$$k \operatorname{sn}\alpha \operatorname{sn}(\alpha+iK') = 1.$$

If we examine the way in which (1) can be obtained by means of the Addition Formulæ, it will appear that, *mutatis mutandis*, sn, cn, dn can be permuted, and corresponding results obtained as follows:—

$$\operatorname{dn}\alpha \operatorname{dn}\beta(\operatorname{dn}^2\alpha - \operatorname{dn}^2\beta)(\operatorname{sn}\gamma \operatorname{cn}\gamma \operatorname{dn}\beta + \operatorname{sn}\beta \operatorname{cn}\beta \operatorname{dn}\gamma)(\operatorname{sn}\alpha \operatorname{cn}\alpha \operatorname{dn}\gamma + \operatorname{sn}\gamma \operatorname{cn}\gamma \operatorname{dn}\alpha)$$
$$+ \operatorname{dn}\beta \operatorname{dn}\gamma(\operatorname{dn}^2\beta - \operatorname{dn}^2\gamma)(\operatorname{sn}\alpha \operatorname{cn}\alpha \operatorname{dn}\gamma + \operatorname{sn}\gamma \operatorname{cn}\gamma \operatorname{dn}\alpha)(\operatorname{sn}\beta \operatorname{cn}\beta \operatorname{dn}\alpha + \operatorname{sn}\alpha \operatorname{cn}\alpha \operatorname{dn}\beta)$$
$$+ \operatorname{dn}\gamma \operatorname{dn}\alpha(\operatorname{dn}^2\gamma - \operatorname{dn}^2\alpha)(\operatorname{sn}\beta \operatorname{cn}\beta \operatorname{dn}\alpha + \operatorname{sn}\alpha \operatorname{cn}\alpha \operatorname{dn}\beta)(\operatorname{sn}\gamma \operatorname{cn}\gamma \operatorname{dn}\beta + \operatorname{sn}\beta \operatorname{cn}\beta \operatorname{dn}\gamma)$$
$$= \operatorname{sn}\alpha \operatorname{cn}\alpha \operatorname{sn}\beta \operatorname{cn}\beta \operatorname{dn}\alpha \operatorname{dn}\beta \operatorname{dn}^2\gamma(\operatorname{dn}^2\alpha - \operatorname{dn}^2\beta + \operatorname{dn}^2\beta - \operatorname{dn}^2\gamma + \operatorname{dn}^2\gamma - \operatorname{dn}^2\alpha) + \ldots$$
$$\ldots + \operatorname{dn}^2\alpha \operatorname{dn}^2\beta(\operatorname{dn}^2\alpha - \operatorname{dn}^2\beta) \operatorname{sn}^2\gamma \operatorname{cn}^2\gamma + \ldots$$
$$= \operatorname{dn}^2\alpha \operatorname{dn}^2\beta(\operatorname{dn}^2\alpha - \operatorname{dn}^2\beta) \operatorname{sn}^2\gamma \operatorname{cn}^2\gamma + \operatorname{dn}^2\beta \operatorname{dn}^2\gamma(\operatorname{dn}^2\beta - \operatorname{dn}^2\gamma) \operatorname{sn}^2\alpha \operatorname{cn}^2\alpha$$
$$+ \operatorname{dn}^2\gamma \operatorname{dn}^2\alpha(\operatorname{dn}^2\gamma - \operatorname{dn}^2\alpha) \operatorname{sn}^2\beta \operatorname{cn}^2\beta.$$

Again,

$$(\mathrm{dn}^2\beta-\mathrm{dn}^2\alpha)\,(\mathrm{dn}^2\alpha-\mathrm{dn}^2\gamma)\,(\mathrm{dn}^2\gamma-\mathrm{dn}^2\beta)$$

$$=-\frac{k^4}{k'^2}\left[\mathrm{dn}^2\alpha\,\mathrm{dn}^2\beta(\mathrm{dn}^2\alpha-\mathrm{dn}^2\beta)\frac{1-\mathrm{dn}^2\gamma}{k^2}\cdot\frac{\mathrm{dn}^2\gamma-k'^2}{k^2}+\ldots\right]$$

$$=-\frac{k^4}{k'^2}\left[\mathrm{dn}^2\alpha\,\mathrm{dn}^2\beta\,(\mathrm{dn}^2\alpha-\mathrm{dn}^2\beta)\,\mathrm{sn}^2\gamma\,\mathrm{cn}^2\gamma+\ldots\right];$$

so that we have

$$\frac{k'^2}{k^4}\cdot\frac{\mathrm{dn}^2\beta-\mathrm{dn}^2\gamma}{\mathrm{sn}\,\gamma\,\mathrm{cn}\,\gamma\,\mathrm{dn}\,\beta+\mathrm{sn}\,\beta\,\mathrm{cn}\,\beta\,\mathrm{dn}\,\gamma}\cdot\frac{\mathrm{dn}^2\gamma-\mathrm{dn}^2\alpha}{\mathrm{sn}\,\alpha\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\gamma+\mathrm{sn}\,\gamma\,\mathrm{cn}\,\gamma\,\mathrm{dn}\,\alpha}$$

$$\times\frac{\mathrm{dn}^2\alpha-\mathrm{dn}^2\beta}{\mathrm{sn}\,\beta\,\mathrm{cn}\,\beta\,\mathrm{dn}\,\alpha+\mathrm{sn}\,\alpha\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\beta}$$

$$=\frac{\mathrm{dn}\,\beta\,\mathrm{dn}\,\gamma\,(\mathrm{dn}^2\beta-\mathrm{dn}^2\gamma)}{\mathrm{sn}\,\gamma\,\mathrm{cn}\,\gamma\,\mathrm{dn}\,\beta+\mathrm{sn}\,\beta\,\mathrm{cn}\,\beta\,\mathrm{dn}\,\gamma}+\ldots;$$

and, since $\dfrac{\mathrm{sn}\,(\alpha-\beta)}{\mathrm{cn}\,(\alpha-\beta)}=\dfrac{\mathrm{sn}^2\alpha-\mathrm{sn}^2\beta}{\mathrm{sn}\,\beta\,\mathrm{cn}\,\beta\,\mathrm{dn}\,\alpha+\mathrm{sn}\,\alpha\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\beta}$,

we have

$$\mathrm{dn}\,\beta\,\mathrm{dn}\,\gamma\,\frac{\mathrm{sn}\,(\beta-\gamma)}{\mathrm{cn}\,(\beta-\gamma)}+\ldots=\frac{k'^2\,\mathrm{sn}\,(\beta-\gamma)\,\mathrm{sn}\,(\gamma-\alpha)\,\mathrm{sn}\,(\alpha-\beta)}{\mathrm{cn}\,(\beta-\gamma)\,\mathrm{cn}\,(\gamma-\alpha)\,\mathrm{cn}\,(\alpha-\beta)}\ldots(3);$$

and, in like manner, or from (3) by changing

$$\mathrm{sn},\ \mathrm{cn},\ \mathrm{dn},\ k^2,\ k'^2,\ \text{ into }\ k\,\mathrm{sn},\ \mathrm{dn},\ \mathrm{cn},\ \frac{1}{k^2},\ -\frac{k'^2}{k^2},$$

we have

$$\mathrm{cn}\,\beta\,\mathrm{cn}\,\gamma\,\frac{\mathrm{sn}\,(\beta-\gamma)}{\mathrm{cn}\,(\beta-\gamma)}+\ldots=-\frac{k'^2\,\mathrm{sn}\,(\beta-\gamma)\,\mathrm{sn}\,(\gamma-\alpha)\,\mathrm{sn}\,(\alpha-\beta)}{\mathrm{cn}\,(\beta-\gamma)\,\mathrm{cn}\,(\gamma-\alpha)\,\mathrm{cn}\,(\alpha-\beta)}\ldots(4);$$

and, increasing in (3) α, &c., by K, we get

$$\mathrm{dn}\,\alpha\,\frac{\mathrm{sn}\,(\beta-\gamma)}{\mathrm{cn}\,(\beta-\gamma)}+\mathrm{dn}\,\beta\,\frac{\mathrm{sn}\,(\gamma-\alpha)}{\mathrm{cn}\,(\gamma-\alpha)}+\mathrm{dn}\,\gamma\,\frac{\mathrm{sn}\,(\alpha-\beta)}{\mathrm{cn}\,(\alpha-\beta)}$$

$$=\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta\,\mathrm{dn}\,\gamma\,\frac{\mathrm{sn}\,(\beta-\gamma)\,\mathrm{sn}\,(\gamma-\alpha)\,\mathrm{sn}\,(\alpha-\beta)}{\mathrm{cn}\,(\beta-\gamma)\,\mathrm{cn}\,(\gamma-\alpha)\,\mathrm{cn}\,(\alpha-\beta)}\ldots(5);$$

increasing in (4) α, &c., by $K+iK'$, we have

$$\mathrm{cn}\,\alpha\,\frac{\mathrm{sn}\,(\beta-\gamma)}{\mathrm{cn}\,(\beta-\gamma)}+\mathrm{cn}\,\beta\,\frac{\mathrm{sn}\,(\gamma-\alpha)}{\mathrm{cn}\,(\gamma-\alpha)}+\mathrm{cn}\,\gamma\,\frac{\mathrm{sn}\,(\alpha-\beta)}{\mathrm{cn}\,(\alpha-\beta)}$$

$$=k^2\,\mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta\,\mathrm{cn}\,\gamma\,\frac{\mathrm{sn}\,(\beta-\gamma)\,\mathrm{sn}\,(\gamma-\alpha)\,\mathrm{sn}\,(\alpha-\beta)}{\mathrm{dn}\,(\beta-\gamma)\,\mathrm{dn}\,(\gamma-\alpha)\,\mathrm{dn}\,(\alpha-\beta)}\ldots(6);$$

It can be easily seen from (1) that

$$\operatorname{sn}(\alpha+\beta+\gamma) = \frac{\operatorname{sn}\alpha\{\operatorname{sn}(\alpha+\beta)\operatorname{sn}\beta-\operatorname{sn}\gamma\operatorname{sn}(\alpha+\gamma)\}}{\operatorname{sn}(\alpha+\gamma)\operatorname{sn}\beta-\operatorname{sn}(\beta+\alpha)\operatorname{sn}\gamma}$$

$$= \frac{\operatorname{sn}\alpha\{\operatorname{sn}\beta(\operatorname{sn}^2\alpha-\operatorname{sn}^2\beta)(\operatorname{sn}\alpha\operatorname{cn}\gamma\operatorname{dn}\gamma-\operatorname{sn}\gamma\operatorname{cn}\alpha\operatorname{dn}\alpha)-\operatorname{sn}\gamma(\operatorname{sn}^2\alpha-\operatorname{sn}^2\gamma)(\operatorname{sn}\alpha\operatorname{cn}\beta\operatorname{dn}\beta-\operatorname{sn}\beta\operatorname{cn}\alpha\operatorname{dn}\alpha\}}{(\operatorname{sn}^2\alpha-\operatorname{sn}^2\gamma)\operatorname{sn}\beta(\operatorname{sn}\alpha\operatorname{cn}\beta\operatorname{dn}\beta-\operatorname{sn}\beta\operatorname{cn}\alpha\operatorname{dn}\alpha)-\operatorname{sn}\gamma(\operatorname{sn}^2\alpha-\operatorname{sn}^2\beta)(\operatorname{sn}\alpha\operatorname{cn}\gamma\operatorname{dn}\gamma-\operatorname{sn}\gamma\operatorname{cn}\alpha\operatorname{dn}\alpha)}$$

$$= -\frac{\operatorname{cn}\alpha\operatorname{dn}\alpha\operatorname{sn}\beta\operatorname{sn}\gamma(\operatorname{sn}^2\beta-\operatorname{sn}^2\gamma)+\operatorname{cn}\beta\operatorname{dn}\beta\operatorname{sn}\gamma\operatorname{sn}\alpha(\operatorname{sn}^2\gamma-\operatorname{sn}^2\alpha)+\operatorname{cn}\gamma\operatorname{dn}\gamma\operatorname{sn}\alpha\operatorname{sn}\beta(\operatorname{sn}^2\alpha-\operatorname{sn}^2\beta)}{\operatorname{cn}\alpha\operatorname{dn}\alpha\operatorname{sn}\alpha(\operatorname{sn}^2\beta-\operatorname{sn}^2\gamma)+\operatorname{cn}\beta\operatorname{dn}\beta\operatorname{sn}\beta(\operatorname{sn}^2\gamma-\operatorname{sn}^2\alpha)+\operatorname{cn}\gamma\operatorname{dn}\gamma\operatorname{sn}\gamma(\operatorname{sn}^2\alpha-\operatorname{sn}^2\beta)}$$

Again, since, ...(7);

$$\operatorname{cn}\alpha\operatorname{sn}(\alpha+\beta)-\operatorname{sn}\alpha\operatorname{dn}\beta\operatorname{cn}(\alpha+\beta) = \frac{\operatorname{cn}\alpha(\operatorname{sn}^2\alpha-\operatorname{sn}^2\beta)-\operatorname{sn}\alpha\operatorname{dn}\beta(\operatorname{sn}\alpha\operatorname{cn}\alpha\operatorname{dn}\beta-\operatorname{sn}\beta\operatorname{cn}\beta\operatorname{dn}\alpha)}{\operatorname{sn}\alpha\operatorname{cn}\beta\operatorname{dn}\beta-\operatorname{sn}\beta\operatorname{cn}\alpha\operatorname{dn}\alpha}$$

$$= \frac{\operatorname{cn}\alpha\operatorname{sn}^2\alpha(1-\operatorname{dn}^2\beta)-\operatorname{sn}\beta(\operatorname{cn}\alpha\operatorname{sn}\beta-\operatorname{sn}\alpha\operatorname{dn}\beta\operatorname{cn}\beta\operatorname{dn}\alpha)}{\operatorname{sn}\alpha\operatorname{cn}\beta\operatorname{dn}\beta-\operatorname{sn}\beta\operatorname{cn}\alpha\operatorname{dn}\alpha}$$

$$= \frac{\operatorname{sn}\beta(k^2\operatorname{cn}\alpha\operatorname{sn}\beta\operatorname{sn}^2\alpha-\operatorname{cn}\alpha\operatorname{sn}\beta+\operatorname{sn}\alpha\operatorname{dn}\beta\operatorname{cn}\beta\operatorname{dn}\alpha)}{\operatorname{sn}\alpha\operatorname{cn}\beta\operatorname{dn}\beta-\operatorname{sn}\beta\operatorname{cn}\alpha\operatorname{dn}\alpha} = \operatorname{sn}\beta\operatorname{dn}\alpha;$$

we have, then,
$$\operatorname{sn}\beta\operatorname{dn}(\alpha+\gamma)\operatorname{cn}(\alpha+\beta+\gamma) = \operatorname{cn}\beta\operatorname{sn}(\alpha+\beta+\gamma)-\operatorname{sn}(\alpha+\gamma)\operatorname{dn}\beta;$$

and
$$\operatorname{sn}\gamma\operatorname{dn}(\alpha+\beta)\operatorname{cn}(\alpha+\beta+\gamma) = \operatorname{cn}\gamma\operatorname{sn}(\alpha+\beta+\gamma)-\operatorname{sn}(\alpha+\beta)\operatorname{dn}\gamma;$$

$$\operatorname{cn}(\alpha+\beta+\gamma) = \frac{\operatorname{sn}(\alpha+\beta)\operatorname{cn}\beta\operatorname{dn}\gamma-\operatorname{sn}(\alpha+\gamma)\operatorname{cn}\gamma\operatorname{dn}\beta}{\operatorname{sn}\beta\operatorname{cn}\gamma\operatorname{dn}(\alpha+\gamma)-\operatorname{sn}\gamma\operatorname{cn}\beta\operatorname{dn}(\alpha+\beta)}$$

$$= \frac{(\operatorname{sn}^2\alpha-\operatorname{sn}^2\beta)\operatorname{cn}\beta\operatorname{dn}\gamma(\operatorname{sn}\alpha\operatorname{cn}\gamma\operatorname{dn}\gamma-\operatorname{sn}\gamma\operatorname{cn}\alpha\operatorname{dn}\alpha)-(\operatorname{sn}^2\alpha-\operatorname{sn}^2\gamma)\operatorname{cn}\gamma\operatorname{dn}\beta(\operatorname{sn}\alpha\operatorname{cn}\beta\operatorname{dn}\beta-\operatorname{sn}\beta\operatorname{cn}\alpha\operatorname{dn}\alpha)}{\operatorname{sn}\beta\operatorname{cn}\gamma(\operatorname{sn}\alpha\operatorname{dn}\alpha\operatorname{cn}\gamma-\operatorname{sn}\gamma\operatorname{dn}\gamma\operatorname{cn}\alpha)(\operatorname{sn}\alpha\operatorname{cn}\beta\operatorname{dn}\beta-\operatorname{sn}\beta\operatorname{cn}\alpha\operatorname{dn}\alpha)}$$
$$-\operatorname{sn}\gamma\operatorname{cn}\beta(\operatorname{sn}\alpha\operatorname{dn}\alpha\operatorname{cn}\beta-\operatorname{sn}\beta\operatorname{dn}\beta\operatorname{cn}\alpha)(\operatorname{sn}\alpha\operatorname{cn}\gamma\operatorname{dn}\gamma-\operatorname{sn}\gamma\operatorname{cn}\alpha\operatorname{dn}\alpha)$$

$$= \frac{\operatorname{sn}\alpha\operatorname{cn}\beta\operatorname{cn}\gamma[\operatorname{dn}^2\gamma(\operatorname{sn}^2\alpha-\operatorname{sn}^2\beta)-\operatorname{dn}^2\beta(\operatorname{sn}^2\alpha-\operatorname{sn}^2\gamma)]+\operatorname{cn}\gamma\operatorname{cn}\alpha\operatorname{dn}\alpha\operatorname{dn}\beta\operatorname{sn}\beta(\operatorname{sn}^2\alpha-\operatorname{sn}^2\gamma)-\operatorname{cn}\alpha\operatorname{cn}\beta\operatorname{sn}\gamma\operatorname{dn}\gamma\operatorname{dn}\alpha(\operatorname{sn}^2\alpha-\operatorname{sn}^2\beta)}{\operatorname{sn}\beta\operatorname{cn}\beta\operatorname{dn}\beta(\operatorname{sn}^2\alpha\operatorname{dn}\alpha\operatorname{cn}^2\gamma-\operatorname{sn}^2\gamma\operatorname{cn}^2\alpha\operatorname{dn}\alpha)+\operatorname{sn}\alpha\operatorname{cn}\alpha\operatorname{dn}^2\alpha(\operatorname{sn}^2\gamma\operatorname{cn}^2\beta-\operatorname{sn}^2\beta\operatorname{cn}^2\gamma)+\operatorname{sn}\gamma\operatorname{cn}\gamma\operatorname{dn}\gamma\operatorname{dn}\alpha(\operatorname{sn}^2\beta\operatorname{cn}^2\alpha-\operatorname{sn}^2\alpha\operatorname{cn}^2\beta)}$$

$$= \frac{\operatorname{sn}\alpha\operatorname{dn}\alpha\operatorname{cn}\beta\operatorname{cn}\gamma(\operatorname{sn}^2\beta-\operatorname{sn}^2\gamma)+\operatorname{sn}\beta\operatorname{dn}\beta\operatorname{cn}\gamma\operatorname{cn}\alpha(\operatorname{sn}^2\gamma-\operatorname{sn}^2\alpha)+\operatorname{sn}\gamma\operatorname{dn}\gamma\operatorname{cn}\alpha\operatorname{cn}\beta(\operatorname{sn}^2\alpha-\operatorname{sn}^2\beta)}{\operatorname{sn}\alpha\operatorname{cn}\alpha\operatorname{dn}\alpha(\operatorname{sn}^2\beta-\operatorname{sn}^2\gamma)+\operatorname{sn}\beta\operatorname{cn}\beta\operatorname{dn}\beta(\operatorname{sn}^2\gamma-\operatorname{sn}^2\alpha)+\operatorname{sn}\gamma\operatorname{cn}\gamma\operatorname{dn}\gamma(\operatorname{sn}^2\alpha-\operatorname{sn}^2\beta)} \quad \text{.....(8),}$$

and, in like manner, or by permuting cn, dn, &c., we have

$$\operatorname{dn}(\alpha+\beta+\gamma) = \frac{\operatorname{sn}\alpha\operatorname{cn}\alpha\operatorname{dn}\beta\operatorname{dn}\gamma(\operatorname{sn}^2\beta-\operatorname{sn}^2\gamma)+\operatorname{sn}\beta\operatorname{cn}\beta\operatorname{dn}\gamma\operatorname{dn}\alpha(\operatorname{sn}^2\gamma-\operatorname{sn}^2\alpha)+\operatorname{sn}\gamma\operatorname{cn}\gamma\operatorname{dn}\alpha\operatorname{dn}\beta(\operatorname{sn}^2\alpha-\operatorname{sn}^2\beta)}{\operatorname{sn}\alpha\operatorname{cn}\alpha\operatorname{dn}\alpha(\operatorname{sn}^2\beta-\operatorname{sn}^2\gamma)+\operatorname{sn}\beta\operatorname{cn}\beta\operatorname{dn}\beta(\operatorname{sn}^2\gamma-\operatorname{sn}^2\alpha)+\operatorname{sn}\gamma\operatorname{cn}\gamma\operatorname{dn}\gamma(\operatorname{sn}^2\alpha-\operatorname{sn}^2\beta)} \quad \text{...(9).}$$

We may remark that we consider the formulæ

$$\operatorname{sn}(\alpha+\beta) = \operatorname{sn}^2\alpha - \operatorname{sn}^2\beta, \qquad (\div)$$

$$\operatorname{cn}(\alpha+\beta) = \operatorname{sn}\alpha\operatorname{cn}\alpha\operatorname{dn}\beta - \operatorname{sn}\beta\operatorname{cn}\beta\operatorname{dn}\alpha, \quad (\div)$$

$$\operatorname{dn}(\alpha+\beta) = \operatorname{sn}\alpha\operatorname{dn}\alpha\operatorname{cn}\beta - \operatorname{sn}\beta\operatorname{dn}\beta\operatorname{cn}\alpha, \quad (\div)$$

where denominator $= \operatorname{sn}\alpha\operatorname{cn}\beta\operatorname{dn}\beta - \operatorname{sn}\beta\operatorname{cn}\alpha\operatorname{dn}\alpha,$

simpler than the elementary formulæ usually given. These do not involve k explicitly, and have numerators and denominators one degree lower than those have.

They become vanishing fractions when

$$\alpha - \beta = 2miK' + 2nK.$$

Thursday, April 6th, 1882.

S. ROBERTS, Esq., F.R.S., President, in the Chair.

Messrs. Buchheim, Muir, and Charles Smith were admitted into the Society.

The following communications were made:—

"The Algebraic Solution of the Modular Equation for the Septic Transformation," G. S. Ely.

"Note on the Condensation of Skew Determinants which are partially zero-axial, and on a Symmetric Determinant connected with Lagrange's Interpolation Problem:" T. Muir.

"On the Addition Equations for the Elliptic and Theta Functions of the Sum of n Arguments:" Rev. M. M. U. Wilkinson.

"On the General Equation of the Second Degree referred to Tetrahedral Coordinates:" Rev. A. J. C. Allen.

"On certain Loci and Envelopes belonging to triangles of given form inscribed and circumscribed to a given triangle:" Prof. Wolstenholme.

"On Binomial Biordinals:" Sir J. Cockle.

"On the Coordinates of a Plane Curve in Space:" H. W. Lloyd Tanner.

"On Polygons circumscribed about a confocal Cubic:" R. A. Roberts.

The following presents were received:—

"Carte-de-Visite Likeness," from F. Scott Haydon, B.A.

"Educational Times" for April, 1882.

"American Journal of Mathematics," Vol iv., No. 2.

"The Mathematical Visitor," Vol. ii., No. 1: from the Editor, Artemas Martin, M.A.

"Bulletin des Sciences Mathématiques et Astronomiques," Tome v, Août, Sept., 1881.

"Beiblätter zu den Annalen der Physik und Chemie," Band vi., Stück 3, No. 3, 1882.

"Bulletin de la Société Mathématique de France," Tome x., No. 1; Paris, 1882.
"Crelle," 92 Band, 1° and 2° Heften; Berlin, 1882.
"Journal de L'École Polytechnique," Cinquantième Cahier, Tome xxxi.; Paris, 1881.
"Atti della R. Accademia dei Lincei," Vol. vi., Fasc. 7°, 8°; Roma, 1882.
"Mittheilungen der Mathematischen Gesellschaft in Hamburg," No. 2 (ausgegeben im März 1882).
(i.) "Lösung der auf die trilineare Verwandtschaft ausgedehnten Projectivitätsproblem."
(ii.) "Elementares Beweis des Feuerbach'schen Satzes," von Dr. Hermann Schubert; Hamburg, 1882.
"Note on Mr. Russell's Paper 'On certain Definite Integrals, No. 10.'" By W. Spottiswoode, P.R.S., from "Proceedings of Royal Society," No. 218, 1882.
"Mathematical Fragments," by the late William Kingdon Clifford, F.R.S.

On the Addition Equations for the Elliptic and Theta Functions of the Sum of n Arguments. *By* Rev. M. M. U. Wilkinson.

[*Read April 6th*, 1882.]

We define as follows:—

$$V(\alpha, \beta) = \frac{\Theta^2 0\, \Theta_1(\alpha-\beta)}{\Theta^2\alpha\, \Theta^2\beta};$$

$$V(\alpha, \beta, \gamma) = \frac{\Theta^5 0\, \Theta_1(\alpha-\beta)\, \Theta_1(\alpha-\gamma)\, \Theta_1(\beta-\gamma)}{\Theta^3\alpha\, \Theta^3\beta\, \Theta^3\gamma};$$

or $$V(\alpha, \beta, \gamma) = \frac{\Theta^3 0\, \Theta_1(\alpha-\beta)\, \Theta_1(\alpha-\gamma)}{\Theta^3\alpha\, \Theta\beta\, \Theta\gamma}\, V(\beta, \gamma);$$

... ... = ...

$$V(\alpha_1, \alpha_2, \alpha_3, \ldots\alpha_n) = \frac{\Theta^n 0 \Theta_1(\alpha_1-\alpha_2)\Theta_1(\alpha_1-\alpha_3)\ldots\Theta_1(\alpha_1-\alpha_n)}{\Theta^n\alpha_1\, \Theta\alpha_2\, \Theta\alpha_3 \ldots \Theta\alpha_n}\, V(\alpha_2, \alpha_3, \ldots\alpha_n);$$

The definitions which we shall now give are by determinants. One line of the determinant only will be written down. The other lines are in every case got by substituting the letters in order.

$$S(\alpha, \beta) = |\ \mathrm{sn}^2\alpha,\ 1\ |;$$

$$C(\alpha, \beta) = |\ \mathrm{sn}\,\alpha\ \mathrm{cn}\,\alpha,\ \mathrm{dn}\,\alpha\ |;$$

$$D(\alpha, \beta) = |\ \mathrm{sn}\,\alpha\ \mathrm{dn}\,\alpha,\ \mathrm{cn}\,\alpha\ |;$$

$$N(\alpha, \beta) = -|\ \mathrm{cn}\,\alpha\ \mathrm{dn}\,\alpha,\ \mathrm{sn}\,\alpha\ |;$$

$$S(\alpha,\beta,\gamma) = |\ \mathrm{sn}^3\alpha,\ \mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha,\ \mathrm{sn}\,\alpha\ |\ ;$$

$$C(\alpha,\beta,\gamma) = -|\ \mathrm{sn}^2\alpha\,\mathrm{cn}\,\alpha,\ \mathrm{sn}\,\alpha\,\mathrm{dn}\,\alpha,\ \mathrm{cn}\,\alpha\ |\ ;$$

$$D(\alpha,\beta,\gamma) = -|\ \mathrm{sn}^2\alpha\,\mathrm{dn}\,\alpha,\ \mathrm{sn}\,\alpha\,\mathrm{cn}\,\alpha,\ \mathrm{dn}\,\alpha\ |\ ;$$

$$N(\alpha,\beta,\gamma) = |\ \mathrm{sn}\,\alpha\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha,\ \mathrm{sn}^2\alpha,\ 1\ |\ ;$$

$$S(\alpha,\beta,\gamma,\delta) = |\ \mathrm{sn}^4\alpha,\ \mathrm{sn}\,\alpha\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha,\ \mathrm{sn}^2\alpha,\ 1\ |\ ;$$

$$C(\alpha,\beta,\gamma,\delta) = -|\ \mathrm{sn}^3\alpha\,\mathrm{cn}\,\alpha,\ \mathrm{sn}^2\alpha\,\mathrm{dn}\,\alpha,\ \mathrm{sn}\,\alpha\,\mathrm{cn}\,\alpha,\ \mathrm{dn}\,\alpha\ |\ ;$$

$$D(\alpha,\beta,\gamma,\delta) = -|\ \mathrm{sn}^3\alpha\,\mathrm{dn}\,\alpha,\ \mathrm{sn}^2\alpha\,\mathrm{cn}\,\alpha,\ \mathrm{sn}\,\alpha\,\mathrm{dn}\,\alpha,\ \mathrm{cn}\,\alpha\ |\ ;$$

$$N(\alpha,\beta,\gamma,\delta) = -|\ \mathrm{sn}^2\alpha\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha,\ \mathrm{sn}^3\alpha,\ \mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha,\ \mathrm{sn}\,\alpha\ |\ ;$$

$$S(\alpha,\beta,\gamma,\delta,\epsilon) = |\ \mathrm{sn}^5\alpha,\ \mathrm{sn}^2\alpha\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha,\ \mathrm{sn}^3\alpha,\ \mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha,\ \mathrm{sn}\,\alpha\ |\ ;$$

$$C(\alpha,\beta,\gamma,\delta,\epsilon) = |\ \mathrm{sn}^4\alpha\,\mathrm{cn}\,\alpha,\ \mathrm{sn}^3\alpha\,\mathrm{dn}\,\alpha,\ \mathrm{sn}^2\alpha\,\mathrm{cn}\,\alpha,\ \mathrm{sn}\,\alpha\,\mathrm{dn}\,\alpha,\ \mathrm{cn}\,\alpha\ |\ ;$$

$$D(\alpha,\beta,\gamma,\delta,\epsilon) = |\ \mathrm{sn}^4\alpha\,\mathrm{dn}\,\alpha,\ \mathrm{sn}^3\alpha\,\mathrm{cn}\,\alpha,\ \mathrm{sn}^2\alpha\,\mathrm{dn}\,\alpha,\ \mathrm{sn}\,\alpha\,\mathrm{cn}\,\alpha,\ \mathrm{dn}\,\alpha\ |\ ;$$

$$N(\alpha,\beta,\gamma,\delta,\epsilon) = |\ \mathrm{sn}^3\alpha\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha,\ \mathrm{sn}^4\alpha,\ \mathrm{sn}\,\alpha\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha,\ \mathrm{sn}^2\alpha,\ 1\ |\ ;$$

$$S(\alpha,\beta,\gamma,\delta,\epsilon,\zeta) = |\ \mathrm{sn}^6\alpha,\ \mathrm{sn}^3\alpha\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha,\ \mathrm{sn}^4\alpha,\ \mathrm{sn}\,\alpha\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha,\ \mathrm{sn}^2\alpha,\ 1\ |\ ;$$

$$C(\alpha,\beta,\gamma,\delta,\epsilon,\zeta) = |\ \mathrm{sn}^5\alpha\,\mathrm{cn}\,\alpha,\ \mathrm{sn}^4\alpha\,\mathrm{dn}\,\alpha,\ \mathrm{sn}^3\alpha\,\mathrm{cn}\,\alpha,\ \mathrm{sn}^2\alpha\,\mathrm{dn}\,\alpha,\ \mathrm{sn}\,\alpha\,\mathrm{cn}\,\alpha,\ \mathrm{dn}\,\alpha\ |\ ;$$

$$D(\alpha,\beta,\gamma,\delta,\epsilon,\zeta) = |\ \mathrm{sn}^5\alpha\,\mathrm{dn}\,\alpha,\ \mathrm{sn}^4\alpha\,\mathrm{cn}\,\alpha,\ \mathrm{sn}^3\alpha\,\mathrm{dn}\,\alpha,\ \mathrm{sn}^2\alpha\,\mathrm{cn}\,\alpha,\ \mathrm{sn}\,\alpha\,\mathrm{dn}\,\alpha,\ \mathrm{cn}\,\alpha\ |\ ;$$

$$N(\alpha,\beta,\gamma,\delta,\epsilon,\zeta) = -|\ \mathrm{sn}^4\alpha\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha,\ \mathrm{sn}^5\alpha,\ \mathrm{sn}^2\alpha\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha,\ \mathrm{sn}^3\alpha,\ \mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha,\ \mathrm{sn}\,\alpha\ |\ ;$$

and so on.

The general definitions will be

$$S(\alpha_1,\alpha_2,\ldots\alpha_{4n+1}) = |\ \mathrm{sn}^{4n+1}\alpha_1,\ \mathrm{sn}^{4n-2}\alpha_1\,\mathrm{cn}\,\alpha_1\,\mathrm{dn}\,\alpha_1,\ \mathrm{sn}^{4n-1}\alpha_1,\ \mathrm{sn}^{4n-4}\alpha_1\,\mathrm{cn}\,\alpha_1\,\mathrm{dn}\,\alpha_1,\ \ldots\,\mathrm{cn}\,\alpha_1\,\mathrm{dn}\,\alpha_1,\ \mathrm{sn}\,\alpha_1\ |\ ;$$

$$C(\alpha_1,\alpha_2,\ldots\alpha_{4n+1}) = |\ \mathrm{sn}^{4n}\alpha_1\,\mathrm{cn}\,\alpha_1,\ \mathrm{sn}^{4n-1}\alpha_1\,\mathrm{dn}\,\alpha_1,\ \ldots\,\mathrm{sn}\,\alpha_1\,\mathrm{dn}\,\alpha_1,\ \mathrm{cn}\,\alpha_1\ |\ ;$$

$$D(\alpha_1,\alpha_2,\ldots\alpha_{4n+1}) = |\ \mathrm{sn}^{4n}\alpha_1\,\mathrm{dn}\,\alpha_1,\ \mathrm{sn}^{4n-1}\alpha_1\,\mathrm{cn}\,\alpha_4,\ \ldots\,\mathrm{sn}\,\alpha_1\,\mathrm{cn}\,\alpha_1,\ \mathrm{dn}\,\alpha_1\ |\ ;$$

$$N(\alpha_1,\alpha_2,\ldots\alpha_{4n+1}) = |\ \mathrm{sn}^{4n-1}\alpha_1\mathrm{cn}\,\alpha_1\,\mathrm{dn}\,\alpha_1,\ \mathrm{sn}^{4n}\alpha_1,\ \ldots\ \ldots\,\mathrm{sn}\,\alpha_1\mathrm{cn}\,\alpha_1\,\mathrm{dn}\,\alpha_1,\ \mathrm{sn}^2\alpha_1,\ 1\ |\ ;$$

$$S(\alpha_1,\alpha_2,\ldots\alpha_{4n+2}) = |\ \mathrm{sn}^{4n+2}\alpha_1,\ \mathrm{sn}^{4n-1}\alpha_1\,\mathrm{cn}\,\alpha_1\,\mathrm{dn}\,\alpha_1,\ \mathrm{sn}^{4n}\alpha_1,\ \ldots\ \ldots\,\mathrm{sn}\,\alpha_1\,\mathrm{cn}\,\alpha_1\,\mathrm{dn}\,\alpha_1,\ \mathrm{sn}^2\alpha_1,\ 1\ |\ ;$$

$$C(\alpha_1,\alpha_2,\ldots\alpha_{4n+2}) = |\ \mathrm{sn}^{4n+1}\alpha_1\,\mathrm{cn}\,\alpha_1,\ \mathrm{sn}^{4n}\alpha_1\,\mathrm{dn}\,\alpha_1,\ \ldots\,\mathrm{sn}\,\alpha_1\,\mathrm{cn}\,\alpha_1,\ \mathrm{dn}\,\alpha_1\ |\ ;$$

$$D(\alpha_1,\alpha_2,\ldots\alpha_{4n+2}) = |\ \mathrm{sn}^{4n+1}\alpha_1\,\mathrm{dn}\,\alpha_1,\ \mathrm{sn}^{4n}\alpha_1\,\mathrm{cn}\,\alpha_1,\ \ldots\,\mathrm{sn}\,\alpha_1\,\mathrm{dn}\,\alpha_1,\ \mathrm{cn}\,\alpha_1\ |\ ;$$

$$N(\alpha_1,\alpha_2,\ldots\alpha_{4n+2}) = -|\ \mathrm{sn}^{4n}\alpha_1\,\mathrm{cn}\,\alpha_1\,\mathrm{dn}\,\alpha_1,\ \mathrm{sn}^{4n+1}\alpha_1,\ \ldots\ \ldots\,\mathrm{sn}^3\alpha_1,\ \mathrm{cn}\,\alpha_1\,\mathrm{dn}\,\alpha_1,\ \mathrm{sn}\,\alpha_1\ |\ ;$$

$$S(a_1, a_2, \dots a_{4n+3}) = |\ \mathrm{sn}^{4n+3} a_1, \mathrm{sn}^{4n} a_1 \mathrm{cn}\, a_1 \mathrm{dn}\, a_1, \mathrm{sn}^{4n+2} a_1, \dots \dots \mathrm{cn}\, a_1 \mathrm{dn}\, a_1, \mathrm{sn}\, a_1 |\ ;$$

$$C(a_1, a_2, \dots a_{4n+3}) = -|\ \mathrm{sn}^{4n+2} a_1 \mathrm{cn}\, a_1, \mathrm{sn}^{4n+1} a_1 \mathrm{dn}\, a_1, \dots \dots \mathrm{sn}\, a_1 \mathrm{dn}\, a_1, \mathrm{cn}\, a_1 |\ ;$$

$$D(a_1, a_2, \dots a_{4n+3}) = -|\ \mathrm{sn}^{4n+2} a_1 \mathrm{dn}\, a_1, \mathrm{sn}^{4n+1} a_1 \mathrm{cn}\, a_1, \dots \dots \mathrm{sn}\, a_1 \mathrm{cn}\, a_1, \mathrm{dn}\, a_1 |\ ;$$

$$N(a_1, a_2, \dots a_{4n+3}) = |\ \mathrm{sn}^{4n+1} a_1 \mathrm{cn}\, a_1 \mathrm{dn}\, a_1, \mathrm{sn}^{4n+2} a_1, \mathrm{sn}^{4n-1} a_1 \mathrm{cn}\, a_1 \mathrm{dn}\, a_1, \dots \mathrm{sn}^2 a_1, 1 |\ ;$$

$$S(a_1, a_2, \dots a_{4n+4}) = |\ \mathrm{sn}^{4n+4} a_1, \mathrm{sn}^{4n+1} a_1 \mathrm{cn}\, a_1 \mathrm{dn}\, a_1, \mathrm{sn}^{4n+2} a_1, \dots \mathrm{sn}^2 a_1, 1 |\ ;$$

$$C(a_1, a_2, \dots a_{4n+4}) = -|\ \mathrm{sn}^{4n+3} a_1 \mathrm{cn}\, a_1, \mathrm{sn}^{4n+2} a_1 \mathrm{dn}\, a_1, \dots \dots \mathrm{sn}\, a_1 \mathrm{cn}\, a_1, \mathrm{dn}\, a_1 |\ ;$$

$$D(a_1, a_2, \dots a_{4n+4}) = -|\ \mathrm{sn}^{4n+3} a_1 \mathrm{dn}\, a_1, \mathrm{sn}^{4n+2} a_1 \mathrm{cn}\, a_1, \dots \dots \mathrm{sn}\, a_1 \mathrm{dn}\, a_1, \mathrm{cn}\, a_1 |\ ;$$

$$N(a_1, a_2, \dots a_{4n+4}) = -|\ \mathrm{sn}^{4n+2} a_1 \mathrm{cn}\, a_1 \mathrm{dn}\, a_1, \mathrm{sn}^{4n+3} a_1, \dots \dots \mathrm{sn}^3 a_1, \mathrm{cn}\, a_1 \mathrm{dn}\, a_1, \mathrm{sn}\, a_1 |\ .$$

These equations will be seen to be satisfied,

$$\begin{aligned} S(\alpha, \beta, \gamma) &= -\mathrm{cn}\,\alpha\ \mathrm{dn}\,\alpha\ \mathrm{sn}\,\beta\ \mathrm{sn}\,\gamma\ S(\beta, \gamma) - \dots \\ &= -\mathrm{sn}^3\alpha\ N(\beta, \gamma) - \dots\ ; \end{aligned}$$

$$\begin{aligned} C(\alpha, \beta, \gamma) &= -\mathrm{cn}\,\alpha\ \mathrm{sn}\,\beta\ \mathrm{sn}\,\gamma\ C(\beta, \gamma) - \dots \\ &= -\mathrm{sn}^2\alpha\ \mathrm{cn}\,\alpha\ D(\beta, \gamma) - \dots\ ; \end{aligned}$$

$$\begin{aligned} D(\alpha, \beta, \gamma) &= -\mathrm{dn}\,\alpha\ \mathrm{sn}\,\beta\ \mathrm{sn}\,\gamma\ D(\beta, \gamma) - \dots \\ &= -\mathrm{sn}^2\alpha\ \mathrm{dn}\,\alpha\ C(\beta, \gamma) - \dots\ ; \end{aligned}$$

$$\begin{aligned} N(\alpha, \beta, \gamma) &= -\mathrm{sn}\,\beta\ \mathrm{sn}\,\gamma\ N(\beta, \gamma) - \dots \\ &= \mathrm{sn}\,\alpha\ \mathrm{cn}\,\alpha\ \mathrm{dn}\,\alpha\ S(\beta, \gamma) + \dots\ ; \end{aligned}$$

$$\begin{aligned} S(\alpha, \beta, \gamma, \delta) &= -\mathrm{sn}\,\beta\ \mathrm{sn}\,\gamma\ \mathrm{sn}\,\delta\ S(\beta, \gamma, \delta) - \dots \\ &= \mathrm{sn}^4\alpha\ N(\beta, \gamma, \delta) + \dots\ ; \end{aligned}$$

$$\begin{aligned} C(\alpha, \beta, \gamma, \delta) &= -\mathrm{dn}\,\alpha\ \mathrm{sn}\,\beta\ \mathrm{sn}\,\gamma\ \mathrm{sn}\,\delta\ C(\beta, \gamma, \delta) - \dots \\ &= \mathrm{sn}^3\alpha\ \mathrm{cn}\,\alpha\ D(\beta, \gamma, \delta) + \dots\ ; \end{aligned}$$

$$\begin{aligned} D(\alpha, \beta, \gamma, \delta) &= -\mathrm{cn}\,\alpha\ \mathrm{sn}\,\beta\ \mathrm{sn}\,\gamma\ \mathrm{sn}\,\delta\ D(\beta, \gamma, \delta) - \dots \\ &= \mathrm{sn}^3\alpha\ \mathrm{dn}\,\alpha\ C(\beta, \gamma, \delta) + \dots\ ; \end{aligned}$$

$$\begin{aligned} N(\alpha, \beta, \gamma, \delta) &= -\mathrm{cn}\,\alpha\ \mathrm{dn}\,\alpha\ \mathrm{sn}\,\beta\ \mathrm{sn}\,\gamma\ \mathrm{sn}\,\delta\ N(\beta, \gamma, \delta) - \dots \\ &= -\mathrm{sn}^2\alpha\ \mathrm{cn}\,\alpha\ \mathrm{dn}\,\alpha\ S(\beta, \gamma, \delta) - \dots\ ; \end{aligned}$$

$$S(\alpha, \beta, \gamma, \delta, \epsilon) = -\operatorname{cn}\alpha \operatorname{dn}\alpha \operatorname{sn}\beta \operatorname{sn}\gamma \operatorname{sn}\delta \operatorname{sn}\epsilon\, S(\beta, \gamma, \delta, \epsilon) - \ldots$$
$$= -\operatorname{sn}^5\alpha\, N(\beta, \gamma, \delta, \epsilon) - \ldots;$$
$$C(\alpha, \beta, \gamma, \delta, \epsilon) = -\operatorname{cn}\alpha \operatorname{sn}\beta \operatorname{sn}\gamma \operatorname{sn}\delta \operatorname{sn}\epsilon\, C(\beta, \gamma, \delta, \epsilon) - \ldots$$
$$= -\operatorname{sn}^4\alpha \operatorname{cn}\alpha\, D(\beta, \gamma, \delta, \epsilon) - \ldots;$$
$$D(\alpha, \beta, \gamma, \delta, \epsilon) = -\operatorname{dn}\alpha \operatorname{sn}\beta \operatorname{sn}\gamma \operatorname{sn}\delta \operatorname{sn}\epsilon\, D(\beta, \gamma, \delta, \epsilon) - \ldots$$
$$= -\operatorname{sn}^4\alpha \operatorname{dn}\alpha\, C(\beta, \gamma, \delta, \epsilon) - \ldots;$$
$$N(\alpha, \beta, \gamma, \delta, \epsilon) = -\operatorname{sn}\beta \operatorname{sn}\gamma \operatorname{sn}\delta \operatorname{sn}\epsilon\, N(\beta, \gamma, \delta, \epsilon) - \ldots$$
$$= \operatorname{sn}^3\alpha \operatorname{cn}\alpha \operatorname{dn}\alpha\, S(\beta, \gamma, \delta, \epsilon) + \ldots.$$

And we shall see that we have, generally,

$$S(a_1, a_2, \ldots a_{2n}) = -\operatorname{sn}a_2 \operatorname{sn}a_3 \ldots \operatorname{sn}a_{2n}\, S(a_2, a_3, \ldots a_{2n}) - \ldots$$
$$= \operatorname{sn}^{2n}a_1\, N(a_2, a_3, \ldots a_{2n}) + \ldots;$$
$$C(a_1, a_2, \ldots a_{2n}) = -\operatorname{dn}a_1 \operatorname{sn}a_2 \operatorname{sn}a_3 \ldots \operatorname{sn}a_{2n}\, C(a_2, a_3, \ldots a_{2n}) - \ldots$$
$$= \operatorname{sn}^{2n-1}a_1 \operatorname{cn}a_1\, D(a_2, a_3, \ldots a_{2n}) + \ldots;$$
$$D(a_1, a_2, \ldots a_{2n}) = -\operatorname{cn}a_1 \operatorname{sn}a_2 \operatorname{sn}a_3 \ldots \operatorname{sn}a_{2n}\, D(a_2, a_3, \ldots a_{2n}) - \ldots$$
$$= \operatorname{sn}^{2n-1}a_1 \operatorname{dn}a_1\, C(a_2, a_3, \ldots a_{2n}) + \ldots;$$
$$N(a_1, a_2, \ldots a_{2n}) = -\operatorname{cn}a_1 \operatorname{dn}a_1 \operatorname{sn}a_2 \operatorname{sn}a_3 \ldots \operatorname{sn}a_{2n}\, N(a_2, a_3, \ldots a_{2n}) - \ldots$$
$$= -\operatorname{sn}^{2n-2}a_1 \operatorname{cn}a_1 \operatorname{dn}a_1\, S(a_2, a_3, \ldots a_{2n}) - \ldots;$$

$$S(a_1, a_2, \ldots a_{2n+1})$$
$$= -\operatorname{cn}a_1 \operatorname{dn}a_1 \operatorname{sn}a_2 \operatorname{sn}a_3 \ldots \operatorname{sn}a_{2n+1}\, S(a_2, a_3, \ldots a_{2n+1}) - \ldots$$
$$= -\operatorname{sn}^{2n+1}a_1\, N(a_2, a_3, \ldots a_{2n+1}) - \ldots;$$
$$C(a_1, a_2, \ldots a_{2n+1}) = -\operatorname{cn}a_1 \operatorname{sn}a_2 \operatorname{sn}a_3 \ldots \operatorname{sn}a_{2n+1}\, C(a_2, a_3, \ldots a_{2n+1}) - \ldots$$
$$= -\operatorname{sn}^{2n}a_1 \operatorname{cn}a_1\, D(a_2, a_3, \ldots a_{2n+1}) - \ldots;$$
$$D(a_1, a_2, \ldots a_{2n+1}) = -\operatorname{dn}a_1 \operatorname{sn}a_2 \operatorname{sn}a_3 \ldots \operatorname{sn}a_{2n+1}\, D(a_2, a_3, \ldots a_{2n+1}) - \ldots$$
$$= -\operatorname{sn}^{2n}a_1 \operatorname{dn}a_1\, C(a_2, a_3, \ldots a_{2n+1}) - \ldots;$$
$$N(a_1, a_2, \ldots a_{2n+1}) = -\operatorname{sn}a_2 \operatorname{sn}a_3 \ldots \operatorname{sn}a_{2n+1}\, N(a_2, a_3, \ldots a_{2n+1}) - \ldots$$
$$= \operatorname{sn}^{2n-1}a_1 \operatorname{cn}a_1 \operatorname{dn}a_1\, S(a_2, a_3, \ldots a_{2n+1}) + \ldots.$$

It will be readily seen that we have

$$S(\alpha, \beta) = \Theta_1(\alpha+\beta)\, V(\alpha, \beta),$$
$$C(\alpha, \beta) = \Theta_2(\alpha+\beta)\, V(\alpha, \beta),$$
$$D(\alpha, \beta) = \Theta_3(\alpha+\beta)\, V(\alpha, \beta),$$
$$N(\alpha, \beta) = \Theta\ (\alpha+\beta)\, V(\alpha, \beta);$$

$$S(\alpha, \beta, \gamma) = -\Theta_1(\alpha+\beta+\gamma)\, V(\alpha, \beta, \gamma),$$
$$C(\alpha, \beta, \gamma) = -\Theta_2(\alpha+\beta+\gamma)\, V(\alpha, \beta, \gamma),$$
$$D(\alpha, \beta, \gamma) = -\Theta_3(\alpha+\beta+\gamma)\, V(\alpha, \beta, \gamma),$$
$$N(\alpha, \beta, \gamma) = -\Theta\ (\alpha+\beta+\gamma)\, V(\alpha, \beta, \gamma).$$

Now, we will suppose that the law established is

$$S(a_1, a_2, \dots a_{4n-3}) = \Theta_1(a_1+a_2+\dots+a_{4n-3})\, V(a_1, a_2, \dots a_{4n-3}),$$
$$C(a_1, a_2, \dots a_{4n-3}) = \Theta_2(a_1+a_2+\dots+a_{4n-3})\, V(a_1, a_2, \dots a_{4n-3}),$$
$$D(a_1, a_2, \dots a_{4n-3}) = \Theta_3(a_1+a_2+\dots+a_{4n-3})\, V(a_1, a_2, \dots a_{4n-3}),$$
$$N(a_1, a_2, \dots a_{4n-3}) = \Theta\,(a_1+a_2+\dots+a_{4n-3})\, V(a_1, a_2, \dots a_{4n-3});$$

$$S(a_1, a_2, \dots a_{4n-2}) = \Theta_1(a_1+a_2+\dots+a_{4n-2})\, V(a_1, a_2, \dots a_{4n-2}),$$
&c. &c. &c.

$$S(a_1, a_2, \dots a_{4n-1}) = -\Theta_1(a_1+a_2+\dots+a_{4n-1})\, V(a_1, a_2, \dots a_{4n-1}),$$
&c. &c. &c.

$$S(a_1, a_2, \dots a_{4n}) = -\Theta_1(a_1+a_2+\dots+a_{4n})\, V(a_1, a_2, \dots a_{4n}),$$
&c. &c. &c.

and show that the same law holds when the number of letters is $(4n+1)$, $(4n+2)$, $(4n+3)$, $(4n+4)$.

An examination of

$$S(a_1, a_2, \dots a_{2n+1})\, S(-a_1, a_2, \dots a_{2n+1})$$

shows that, when considered as a function of a_1, it is a rational integral algebraic function of $\mathrm{sn}^2 a_1$ of the degree $(2n+1)$, and can be resolved into its factors, since $2n$ values of $\mathrm{sn}^2 a_1$ are known which make the expression vanish, and the coefficients of $\mathrm{sn}^{4n+2} a_1$ and $\mathrm{sn}^0 a_1$ are known.

We have then $S(a_1, a_2, \dots a_{2n+1})\, S(-a_1, a_2, \dots a_{2n+1})$

$$= -N^2(a_2, a_3, \dots a_{2n+1})\, S(a_1, a_2)\, S(a_1, a_3) \dots S(a_1, a_{2n+1}) \times \left\{ \mathrm{sn}^2 a_1 - \frac{S^2(a_2, a_3, \dots a_{2n+1})}{N^2(a_2, a_3, \dots a_{2n+1})} \right\}$$

$$= -N^2(a_2, a_3, \dots a_{2n+1})\, S(a_1, a_2)\, S(a_1, a_3) \dots S(a_1, a_{2n+1}) \times S(a_1, a_2+a_3+\dots+a_{2n+1})$$

$$= \Theta^2(a_2+a_3+\dots+a_{2n+1})\, V^2(a_2, a_3, \dots a_{2n+1}) \times \frac{\Theta^4 0\, \Theta_1(a_1-a_2)\, \Theta_1(a_1+a_2) \dots \Theta_1(a_1+a_{2n-1})\, \Theta_1(a_1-a_{2n+1})}{\Theta^{4n} a_1\, \Theta^2 a_2\, \Theta^2 a_3 \dots \Theta^2 a_{2n+1}} \times \frac{\Theta^2 0\, \Theta_1(a_1+a_2+\dots+a_{2n+1})\, \Theta_1(-a_1+a_2+\dots+a_{2n+1})}{\Theta^2 a_1\, \Theta^2(a_2+a_3+\dots+a_{2n+1})}$$

$$= \Theta_1(a_1+a_2+\dots+a_{2n+1})\, \Theta_1(-a_1+a_2+\dots+a_{2n+1}) \times V(a_1, a_2, \dots a_{2n+1})\, V(-a_1, a_2, \dots a_{2n+1}).$$

In like manner, we have

$$C(a_1, a_2, \dots a_{2n+1})\, C(-a_1, a_2, \dots a_{2n+1})$$
$$= -D^2(a_2, a_3, \dots a_{2n+1})\, S(a_1, a_2)\, S(a_1, a_3) \dots S(a_1, a_{2n+1}) \times \left\{ \mathrm{sn}^2 a_1 - \frac{C^2(a_2, a_3, \dots a_{2n+1})}{D^2(a_2, a_3, \dots a_{2n+1})} \right\}$$

$$= - D^2(a_2, a_3, \dots a_{2n+1})\, S(a_1, a_2) \dots S(a_1, a_{2n+1})$$
$$\times S(a_1, K + a_2 + a_3 + \dots + a_{2n+1})$$
$$= \Theta_3^2(a_2 + a_3 + \dots + a_{2n+1})\, V^2(a_2, a_3, \dots a_{2n+1})$$
$$\times \frac{\Theta^{4n}0\, \Theta_1(a_1 - a_2)\, \Theta_1(a_1 + a_2) \dots \Theta_1(a_1 - a_{2n+1})\, \Theta_1(a_1 + a_{2n+1})}{\Theta^{4n} a_1 \Theta^2 a_2 \Theta^2 a_3 \dots \Theta^2 a_{2n+1}}$$
$$\times \frac{\Theta^2 0\, \Theta_2(a_1 + a_2 + \dots + a_{2n+1})\, \Theta_2(-a_1 + a_2 + \dots + a_{2n+1})}{\Theta^2 a_1\, \Theta_3^2(a_2 + a_3 + \dots + a_{2n+1})}$$
$$= \Theta_2(a_1 + a_2 + \dots + a_{2n+1})\, \Theta_2(-a_1 + a_2 + \dots + a_{2n+1})$$
$$\times V(a_1, a_2, \dots a_{2n+1})\, V(-a_1, a_2, \dots a_{2n+1}).$$

In like manner, we have,

$$D(a_1, a_2, \dots a_{2n+1})\, D(-a_1, a_2, \dots a_{2n+1})$$
$$= - k^2 C^2(a_2, a_3, \dots a_{2n+1})\, S(a_1, a_2)\, S(a_1, a_3) \dots S(a_1, a_{2n+1})$$
$$\times \left\{ \operatorname{sn}^2 a_1 - \frac{D^2(a_2, a_3, \dots a_{2n+1})}{k^2 C^2(a_2, a_3, \dots a_{2n+1})} \right\}$$
$$= + C^2(a_2, a_3, \dots a_{2n+1})\, S(a_1, a_2)\, S(a_1, a_3) \dots S(a_1, a_{2n+1})$$
$$\times \left\{ \frac{1 - k^2 \operatorname{sn}^2 a_1 \operatorname{sn}^2(K + a_2 + a_3 + \dots + a_{2n+1})}{\operatorname{sn}^2(K + a_2 + a_3 + \dots + a_{2n+1})} \right\}$$
$$= \Theta_2^2(a_2 + a_3 + \dots + a_{2n+1})\, V^2(a_2, a_3, \dots a_{2n+1})$$
$$\times \frac{\Theta^{4n}0\, \Theta_1(a_1 - a_2)\, \Theta_1(a_1 + a_2) \dots \Theta_1(a_1 - a_{2n+1})\, \Theta_1(a_1 + a_{2n+1})}{\Theta^{4n} a_1 \Theta^2 a_2 \Theta^2 a_3 \dots \Theta^2 a_{2n+1}}$$
$$\times \frac{\Theta^2 0\, \Theta_3(a_1 + a_2 + \dots + a_{2n+1})\, \Theta_3(-a_1 + a_2 + \dots + a_{2n+1})}{\Theta^2 a_1\, \Theta_2^2(a_2 + a_3 + \dots + a_{2n+1})}$$
$$= \Theta_3(a_1 + a_2 + \dots + a_{2n+1})\, \Theta_3(-a_1 + a_2 + \dots + a_{2n+1})$$
$$V(a_1, a_2, \dots a_{2n+1})\, V(-a_1, a_2, \dots a_{2n+1}).$$

In like manner, we have

$$N(a_1, a_2, \dots a_{2n+1})\, N(-a_1, a_2, a_3, \dots a_{2n+1})$$
$$= - k^2 S^2(a_2, a_3, \dots a_{2n+1})\, S(a_1, a_2) \dots S(a_1, a_{2n+1})$$
$$\times \left\{ \operatorname{sn}^2 a_1 - \frac{N^2(a_2, a_3, \dots a_{2n+1})}{k^2 S^2(a_2, a_3, \dots a_{2n+1})} \right\}$$
$$= S^2(a_2, a_3, \dots a_{2n+1})\, S(a_1, a_2) \dots S(a_1, a_{2n+1})$$
$$\times \left\{ \frac{1 - k^2 \operatorname{sn}^2 a_1 \operatorname{sn}^2(a_2 + a_3 + \dots + a_{2n+1})}{\operatorname{sn}^2(a_2 + a_3 + \dots + a_{2n+1})} \right\}$$
$$= \Theta_1^2(a_2 + a_3 + \dots + a_{2n+1})\, V^2(a_2, a_3, \dots a_{2n+1})$$
$$\times \frac{\Theta^{4n}0\, \Theta_1(a_1 - a_2)\, \Theta_1(a_1 + a_2) \dots \Theta_1(a_1 - a_{2n+1})\, \Theta_1(a_1 + a_{2n+1})}{\Theta^{4n} a_1 \Theta^2 a_2 \Theta^2 a_3 \dots \Theta^2 a_{2n+1}}$$
$$\times \frac{\Theta^2 0\, \Theta(a_1 + a_2 + \dots + a_{2n+1})\, \Theta(-a_1 + a_2 + \dots + a_{2n+1})}{\Theta^2 a\, \Theta_1^2\ a_2 + a_3 + \dots + a_{2n+1})}$$
$$= \Theta(a_1 + a_2 + \dots + a_{2n+1})\, \Theta(-a_1 + a_2 + \dots + a_{2n+1})$$
$$\times V(a_1, a_2, \dots a_{2n+1})\, V(-a_1, a_2, \dots a_{2n+1}).$$

Now, from the system of equations which give

$$x_0x_1,\ x_1x_2,\ x_2x_3,\ \dots\ x_{2n-1}x_{2n},\ x_{2n}x_0,$$

we can get $x_0^2 x_1^2 x_2^2 \dots x_{2n}^2$, and also $x_1x_2 \dots x_{2n-1}x_{2n}$, and so x_0^2 can be got. This shows that we have

$$S(a_1, a_2, \dots a_{2n+1}) = (-1)^n \Theta_1 (a_1+a_2+\dots+a_{2n+1})\ V(a_1, a_2, a_3, \dots a_{2n+1}),$$

$$C(a_1, a_2, \dots a_{2n+1}) = (-1)^n \Theta_2 (a_1+a_2+\dots+a_{2n+1})\ V(a_1, a_2, a_3, \dots a_{2n+1}),$$

$$D(a_1, a_2, \dots a_{2n+1}) = (-1)^n \Theta_3 (a_1+a_2+\dots+a_{2n+1})\ V(a_1, a_2, \dots a_{2n+1}),$$

$$N(a_1, a_2, \dots a_{2n+1}) = (-1)^n \Theta\ (a_1+a_2+\dots+a_{2n+1})\ V(a_1, a_2, \dots a_{2n+1}).$$

The sign being determined by putting $a_1 = 0$.

Before taking the case when the number of letters is even, we will notice that, if

$$\begin{aligned}&(\mathrm{sn}^4 a + C_2 \mathrm{sn}^2 a + C_3 \mathrm{sn}\, a\, \mathrm{cn}\, a\, \mathrm{dn}\, a + C_4)\\ &\qquad \times (\mathrm{sn}^4 a + C_2 \mathrm{sn}^2 a - C_3 \mathrm{sn}\, a\, \mathrm{cn}\, a\, \mathrm{dn}\, a + C_4)\\ &= (\mathrm{sn}^4 a + c_2 \mathrm{sn}^2 a + c_3 \mathrm{sn}\, a\, \mathrm{cn}\, a\, \mathrm{dn}\, a + c_4)\\ &\qquad \times (\mathrm{sn}^4 a + c_2 \mathrm{sn}^2 a - c_3 \mathrm{sn}\, a\, \mathrm{cn}\, a\, \mathrm{dn}\, a + c_4);\end{aligned}$$

for all values of a, we must have

$$\begin{aligned}2C_2 - C_3^2 k^2 &= 2c_2 - c_3^2 k^2,\\ C_2^2 + 2C_4 + (1+k^2) C_3^2 &= c_2^2 + 2c_4 + (1+k^2) c_3^2,\\ 2C_2 C_4 - C_3^2 &= 2c_2 c_4 - c_3^2,\\ C_4^2 &= c_4^2;\end{aligned}$$

whence $(C_2+C_4)^2 + 2(C_2+C_4) = (c_2+c_4)^2 + 2(c_2+c_4)$,

and $C_2 + C_4 = c_2 + c_4$, or $-2 - c_2 - c_4$;

and also we have $2C_2(1-k^2C_4) = 2c_2(1-k^2c_4)$,

so that we must have, unless special relations connect c_2, c_4,

$$C_2 = c_2, \quad \text{and} \quad C_4 = c_4,$$

and so, again, $C_3^2 = c_3^2$.

We can now assert the identity

$$S^2(a, \beta)\, S(a, \beta+\gamma+\delta)\, S(a, -\beta+\gamma+\delta) = \frac{[S^2(\gamma,\delta)N^2(a,\beta) - S^2(a,\beta)N^2(\gamma,\delta)][S^2(\gamma,\delta)N^2(-a,\beta) - S^2(-a,\beta)N^2(\gamma,\delta)]}{[S^2(\gamma,\delta)\, k^2 \mathrm{sn}^2\beta - N^2(\gamma,\delta)]^2}$$

for both members of this equation are of the fourth degree in $\mathrm{sn}^2 a$, $S^2(a, \beta)$ is a factor of both, for

$$\begin{aligned}N^2(a, \beta)\, N^2(-a, \beta) &= (\mathrm{sn}^2 a\, \mathrm{cn}^2\beta\, \mathrm{dn}^2\beta - \mathrm{sn}^2\beta\, \mathrm{cn}^2 a\, \mathrm{dn}^2 a)^2\\ &= S^2(a, \beta)\,(1 - k^2 \mathrm{sn}^2 a\, \mathrm{sn}^2\beta)^2,\end{aligned}$$

and either $\mathrm{sn}^2 a = \mathrm{sn}^2(\beta+\gamma+\delta)$, or $\mathrm{sn}^2 a = \mathrm{sn}^2(-\beta+\gamma+\delta)$,

makes both members of the equation vanish, and the coefficient of $\mathrm{sn}^8 a$ in each is the same.

As before, we have

$$S(a_1, a_2, \dots a_{2n})\, S(-a_1, a_2, \dots a_{2n})$$

$$= N^2(a_2, a_3, \dots a_{2n})\, S(a_1, a_2)\, S(a_1, a_3) \dots S(a_1, a_{2n}) \times \left\{ \mathrm{sn}^2 a_1 - \frac{S^2(a_2, a_3, \dots a_{2n})}{N^2(a_2, a_3, \dots a_{2n})} \right\}$$

$$= N^2(a_2, a_3, \dots a_{2n})\, S(a_1, a_2)\, S(a_1, a_3) \dots S(a_1, a_{2n})\, S(a_1, a_2 + a_3 + \dots + a_{2n})$$

$$= \Theta^2(a_2 + a_3 + \dots + a_{2n})\, V^2(a_2, a_3, \dots a_{2n}) \times \frac{\Theta^{4n-2}0\, \Theta_1(a_1 - a_2)\, \Theta_1(a_1 + a_2) \dots \Theta_1(a_1 - a_{2n})\, \Theta_1(a_1 + a_{2n})}{\Theta^{4n-2}a_1 \Theta^2 a_2 \Theta^2 a_3 \dots \Theta^2 a_{2n}}$$

$$\times \frac{\Theta^2 0\, \Theta_1(a_1 + a_2 + \dots + a_{2n})\, \Theta_1(-a_1 + a_2 + \dots + a_{2n})}{\Theta^2 a_1 \Theta^2(a_2 + a_3 + \dots + a_{2n})}$$

$$= \Theta_1(a_1 + a_2 + \dots + a_{2n})\, \Theta_1(-a_1 + a_2 + \dots + a_{2n}) \times V(a_1, a_2, \dots a_{2n})\, V(-a_1, a_2, \dots a_{2n}).$$

In like manner, we have

$$C(a_1, a_2, \dots a_{2n})\, C(-a_1, a_2, \dots a_{2n})$$

$$= D^2(a_2, a_3, \dots a_{2n})\, S(a_1, a_2)\, S(a_1, a_3) \dots S(a_1, a_{2n}) \times \left\{ \mathrm{sn}^2 a_1 - \frac{C^2(a_2, a_3, \dots a_{2n})}{D^2(a_2, a_3, \dots a_{2n})} \right\}$$

$$= D^2(a_2, a_3, \dots a_{2n})\, S(a_1, a_2)\, S(a_1, a_3) \dots S(a_1, a_{2n}) \times S(a_1, K + a_2 + a_3 + \dots + a_{2n})$$

$$= \Theta_3^2(a_2 + a_3 + \dots + a_{2n})\, V^2(a_2, a_3, \dots a_{2n}) \times \frac{\Theta^{4n-2}0\, \Theta_1(a_1 - a_2)\, \Theta_1(a_1 + a_2) \dots \Theta_1(a_1 - a_{2n})\, \Theta_1(a_1 + a_{2n})}{\Theta^{4n-2}a_1 \Theta^2 a_2 \Theta^2 a_3 \dots \Theta^2 a_{2n}}$$

$$\times \frac{\Theta^2 0\, \Theta_2(a_1 + a_2 + \dots + a_{2n})\, \Theta_2(-a_1 + a_2 + \dots + a_{2n})}{\Theta^2 a_1 \Theta^2(a_2 + a_3 + \dots + a_{2n})}$$

$$= \Theta_2(a_1 + a_2 + \dots + a_{2n})\, \Theta_2(-a_1 + a_2 + \dots + a_{2n}) \times V(a_1, a_2, \dots a_{2n})\, V(-a_1, a_2, \dots a_{2n}).$$

In like manner, we have

$$D(a_1, a_2, \dots a_{2n})\, D(-a_1, a_2, \dots a_{2n})$$

$$= k^2 C^2(a_2, a_3, \dots a_{2n})\, S(a_1, a_2)\, S(a_1, a_3) \dots S(a_1, a_{2n}) \times \left\{ \mathrm{sn}^2 a_1 - \frac{D^2(a_2, a_3, \dots a_{2n})}{k^2 C^2(a_2, a_3, \dots a_{2n})} \right\}$$

$$= - C^2(a_2, a_3, \dots a_{2n})\, S(a_1, a_2)\, S(a_1, a_3) \dots S(a_1, a_{2n}) \times \left\{ \frac{1 - k^2 \operatorname{sn}^2 a_1 \operatorname{sn}^2 (K + a_2 + a_3 + \dots + a_{2n})}{\operatorname{sn}^2 (K + a_2 + a_3 + \dots + a_{2n})} \right\}$$

$$= \Theta_2^2 (a_2 + a_3 + \dots + a_{2n})\, V^2 (a_2, a_3, \dots a_{2n}) \times \frac{\Theta^{4n-2} 0\, \Theta_1 (a_1 - a_2)\, \Theta_1 (a_1 + a_2) \dots \Theta_1 (a_1 - a_{2n})\, \Theta_1 (a_1 + a_{2n})}{\Theta^{4n-2} a_1 \Theta^2 a_2 \Theta^2 a_3 \dots \Theta^2 a_{2n}} \times \frac{\Theta^2 0\, \Theta_3 (a_1 + a_2 + \dots + a_{2n})\, \Theta_3 (-a_1 + a_2 + \dots + a_{2n})}{\Theta^2 a_1 \Theta_2^2 (a_2 + a_3 + \dots + a_{2n})}$$

$$= \Theta_3 (a_1 + a_2 + \dots + a_{2n})\, \Theta_3 (-a_1 + a_2 + \dots + a_{2n}) \times V(a_1, a_2, \dots a_{2n})\, V(-a_1, a_2, \dots a_{2n}).$$

In like manner, we have

$$N(a_1, a_2, \dots a_{2n})\, N(-a_1, a_2, \dots a_{2n})$$

$$= k^2 S^2 (a_2, a_3, \dots a_{2n})\, S(a_1, a_2)\, S(a_1, a_3) \dots S(a_1, a_{2n}) \times \left\{ \operatorname{sn}^2 a_1 - \frac{N^2 (a_2, a_3, \dots a_{2n})}{k^2 S^2 (a_2, a_3, \dots a_{2n})} \right\}$$

$$= - S^2 (a_2, a_3, \dots a_{2n})\, S(a_1, a_2)\, S(a_1, a_3) \dots S(a_1, a_{2n}) \times \left\{ \frac{1 - k^2 \operatorname{sn}^2 a_1 \operatorname{sn}^2 (a_2 + a_3 + \dots + a_{2n})}{\operatorname{sn}^2 (a_2 + a_3 + \dots + a_{2n})} \right\}$$

$$= \Theta_1^2 (a_2 + a_3 + \dots + a_{2n})\, V^2 (a_2, a_3, \dots a_{2n}) \times \frac{\Theta^{4n-2} 0\, \Theta_1 (a_1 - a_2)\, \Theta_1 (a_1 + a_2) \dots \Theta_1 (a_1 - a_{2n})\, \Theta_1 (a_1 + a_{2n})}{\Theta^{4n-2} a_1 \Theta^2 a_2 \Theta^2 a_3 \dots \Theta^2 a_{2n}} \times \frac{\Theta^2 0\, \Theta (a_1 + a_2 + \dots + a_{2n})\, \Theta (-a_1 + a_2 + \dots + a_{2n})}{\Theta^2 a_1 \Theta_1^2 (a_2 + a_3 + \dots + a_{2n})}$$

$$= \Theta (a_1 + a_2 + \dots + a_{2n})\, \Theta (-a_1 + a_2 + \dots + a_{2n}) \times V(a_1, a_2, \dots a_{2n})\, V(-a_1, a_2, \dots a_{2n}).$$

Now, if we examine

$$S(a_1, a_2, a_3, \dots a_{2n})\, S(-a_1, -a_2, a_3, \dots a_{2n}),$$

we see that it can be written in the form

$$A_1 + A_2 \operatorname{sn} a_1 \operatorname{cn} a_1 \operatorname{dn} a_1;$$

where A_1, A_2 only contain a_1 through even powers of $\operatorname{sn} a_1$; also we shall have $S(-a_1, a_2, a_3, \dots a_{2n})\, S(a_1, -a_2, a_3, \dots a_{2n})$ in the form

$$A_1 - A_2 \operatorname{sn} a_1 \operatorname{cn} a_1 \operatorname{dn} a_1.$$

Further, both A_1 and A_2 must evidently contain as factors,

$$(a_1, a_3)\, S(a_1, a_4) \dots S(a_1, a_{2n}).$$

Hence it is clear that we may write

$$S(a_1, a_2, a_3, \dots a_{2n})\, S(-a_1, -a_2, a_3, \dots a_{2n})$$
$$= N(a_2, a_3, \dots a_{2n})\, N(-a_2, a_3, \dots a_{2n})\, S(a_1, a_3)\, S(a_1, a_4) \dots S(a_1, a_{2n})$$
$$\times \{\mathrm{sn}^4 a_1 + C_2 \mathrm{sn}^2 a_1 + C_3 \mathrm{sn}\, a_1 \mathrm{cn}\, a_1 \mathrm{dn}\, a_1 + C_4\},$$

and
$$S(-a_1, a_2, a_3, \dots a_{2n})\, S(a_1, -a_2, a_3, \dots a_{2n})$$
$$= N(a_2, a_3, \dots a_{2n})\, N(-a_2, a_3, \dots a_{2n})\, S(a_1, a_3)\, S(a_1, a_4) \dots S(a_1, a_{2n})$$
$$\times \{\mathrm{sn}^4 a_1 + C_2 \mathrm{sn}^2 a_1 - C_3 \mathrm{sn}\, a_1 \mathrm{cn}\, a_1 \mathrm{dn}\, a_1 + C_4\}.$$

It follows from this that

$$\{\mathrm{sn}^4 a_1 + C_2 \mathrm{sn}^2 a_1 + C_3 \mathrm{sn}\, a_1 \mathrm{cn}\, a_1 \mathrm{dn}\, a_1 + C_4\}$$
$$\{\mathrm{sn}^4 a_1 + C_2 \mathrm{sn}^2 a_1 - C_3 \mathrm{sn}\, a_1 \mathrm{cn}\, a_1 \mathrm{dn}\, a_1 + C_4\}$$
$$= S^2(a_1, a_2)\, S(a_1, a_2 + a_3 + \dots + a_{2n})\, S(a_1, -a_2 + a_3 + \dots + a_{2n})$$
$$= [S^2(a_3, a_4 + \dots + a_{2n})\, N^2(a_1, a_2) - S^2(a_1, a_2)\, N^2(a_3, a_4 + \dots + a_{2n})]$$
$$\times [S^2(a_3, a_4 + \dots + a_{2n})\, N^2(-a_1, a_2) - S^2(-a_1, a_2)\, N^2(a_3, a_4 + \dots + a_{2n})]$$
$$\div [S^2(a_3, a_4 + \dots + a_{2n})\, k^2 \mathrm{sn}^2 a_2 - N^2(a_3, a_4 + \dots + a_{2n})],$$

and, from what we have shown before, we see that, to a factor *près* (which factor does not contain a_1),

$$\mathrm{sn}^4 a_1 + C_2 \mathrm{sn}^2 a_1 + C_3 \mathrm{sn}\, a_1 \mathrm{cn}\, a_1 \mathrm{dn}\, a_1 + C_4$$
$$= S^2(a_3, a_4 + \dots + a_{2n})\, N^2(a_1, a_2) - S^2(a_1, a_2)\, N^2(a_3, a_4 + \dots + a_{2n}).$$

And so we have

$$S(a_1, a_2, a_3, \dots a_{2n})\, S(-a_1, -a_2, a_3, \dots a_{2n})$$
$$= N(a_2, a_3, \dots a_{2n})\, N(-a_2, a_3, \dots a_{2n})\, S(a_1, a_3)\, S(a_1, a_4) \dots S(a_1, a_{2n})$$
$$\times \frac{S^2(a_3, a_4 + \dots + a_{2n})\, N^2(a_1, a_2) - S^2(a_1, a_2)\, N^2(a_3, a_4 + \dots + a_{2n})}{k^2 \mathrm{sn}^2 a_2\, S^2(a_3, a_4 + \dots + a_{2n}) - N^2(a_3, a_4 + \dots + a_{2n})}$$
$$= N(a_2, a_3, \dots a_{2n})\, N(-a_2, a_3, \dots a_{2n})\, S(a_1, a_3)\, S(a_1, a_4) \dots S(a_1, a_{2n})$$
$$\times \frac{\mathrm{sn}^2(a_3 + a_4 + \dots + a_{2n}) - \mathrm{sn}^2(a_1 + a_2)}{k^2 \mathrm{sn}^2 a_2 \mathrm{sn}^2(a_3 + a_4 + \dots + a_{2n}) - 1} \cdot N^2(a_1, a_2)$$
$$= -\Theta(a_2 + a_3 + \dots + a_{2n})\, \Theta(-a_2 + a_3 + \dots + a_{2n})$$
$$\times V(a_2, a_3, \dots a_{2n})\, V(-a_2, a_3, \dots a_{2n})$$
$$\times \frac{\Theta^{4n-4}0\, \Theta_1(a_1 - a_3)\, \Theta_1(a_1 + a_3) \dots \Theta_1(a_1 - a_{2n})\, \Theta_1(a_1 + a_{2n})}{\Theta^{4n-4} a_1 \Theta^2 a_3 \Theta^2 a_4 \dots \Theta^2 a_{2n}}$$
$$\times \frac{\Theta^2 0\, \Theta_1(a_1 + a_2 + \dots + a_{2n})\, \Theta_1(-a_1 - a_2 + a_3 + \dots + a_{2n})}{\Theta^2(a_1 + a_2)\, \Theta^2(a_3 + a_4 + \dots + a_{2n})}$$
$$\times \frac{\Theta^2 a_2\, \Theta^2(a_3 + a_4 + \dots + a_{2n})}{\Theta^2 0\, \Theta(a_2 + a_3 + \dots + a_{2n})\, \Theta(-a_2 + a_3 + \dots + a_{2n})}$$
$$\times \frac{\Theta^4 0\, \Theta^2(a_1 + a_2)\, \Theta_1^2(a_1 - a_2)}{\Theta^4 a_1 \Theta^4 a_2}$$

$$= -V(a_2, a_3, \dots a_{2n})\, V(-a_2, a_3, \dots a_{2n})$$
$$\times \frac{\Theta^{4n}0\ \Theta_1(a_1-a_3)\ \Theta_1(a_1+a_3) \dots \Theta_1(a_1-a_{2n})\ \Theta_1(a_1+a_{2n})}{\Theta^{4n}a_1\, \Theta^2 a_2 \Theta^2 a_3 \dots \Theta^2 a_{2n}}$$
$$\times \Theta_1(a_1+a_2+\dots+a_{2n})\, \Theta_1(-a_1-a_2+a_3+\dots+a_{2n})\, .\, \Theta_1^2(a_1-a_2)$$
$$= \Theta_1(a_1+a_2+\dots+a_{2n})\, \Theta_1(-a_1-a_2+a_3+\dots+a_{2n})$$
$$\times V(a_1, a_2, \dots a_{2n})\, V(-a_1, -a_2, a_3, \dots a_{2n}).$$

Now, if we have given x_0x_1, x_0x_2, x_1x_2, we can find x_0^2. So we have

$$S(a_1, a_2, \dots a_{2n}) = (-1)^{n-1}\, \Theta_1(a_1+a_2+\dots+a_{2n})\, V(a_1, a_2, \dots a_{2n}),$$

the sign being determined by putting $a_1 = 0$. And if we examine

$$C(a_1, a_2, \dots a_{2n})\, C(-a_1, -a_2, a_3, \dots a_{2n}),$$

we see that it can be written in the form

$$A_3 + A_4 \operatorname{sn} a_1 \operatorname{cn} a_1 \operatorname{dn} a_1,$$

where A_3, A_4 only contain a_1 through even powers of $\operatorname{sn} a_1$. Also we have

$$C(-a_1, a_2, \dots a_{2n})\, C(a_1, -a_2, a_3, \dots a_{2n})$$
$$= A_3 - A_4 \operatorname{sn} a_1 \operatorname{cn} a_1 \operatorname{dn} a_1.$$

Further, both A_3 and A_4 must evidently contain as factors

$$S(a_1, a_3)\, S(a_1, a_4) \dots S(a_1, a_{2n}).$$

Hence it is clear that we may write

$$C(a_1, a_2, \dots a_{2n})\, C(-a_1, -a_2, a_3, \dots a_{2n})$$
$$= D(a_2, a_3, \dots a_{2n})\, D(-a_2, a_3, \dots a_{2n})\, S(a_1, a_3)\, S(a_1, a_4) \dots S(a_1, a_{2n})$$
$$\times \{\operatorname{sn}^4 a_1 + C_6 \operatorname{sn}^2 a_1 + C_7 \operatorname{sn} a_1 \operatorname{cn} a_1 \operatorname{dn} a_1 + C_8\};$$

and

$$C(-a_1, a_2, \dots a_{2n})\, C(a_1, -a_2, a_3, \dots a_{2n})$$
$$= D(a_2, a_3, \dots a_{2n})\, D(-a_2, a_3, \dots a_{2n})\, S(a_1, a_3)\, S(a_1, a_4) \dots S(a_1, a_{2n})$$
$$\times \{\operatorname{sn}^4 a_1 + C_6 \operatorname{sn}^2 a_1 - C_7 \operatorname{sn} a_1 \operatorname{cn} a_1 \operatorname{dn} a_1 + C_8\}.$$

It follows from this that

$$\{\operatorname{sn}^4 a_1 + C_6 \operatorname{sn}^2 a_1 + C_7 \operatorname{sn} a_1 \operatorname{cn} a_1 \operatorname{dn} a_1 + C_8\}$$
$$\times \{\operatorname{sn}^4 a_1 + C_6 \operatorname{sn}^2 a_1 - C_7 \operatorname{sn} a_1 \operatorname{cn} a_1 \operatorname{dn} a_1 + C_8\}$$
$$= S^2(a_1, a_2)\, S(a_1, K+a_2+a_3+\dots+a_{2n})\, S(a_1, K-a_2+a_3+\dots+a_{2n})$$
$$= [S^2(a_3, K+a_4+\dots+a_{2n}) N^2(a_1, a_2) - S^2(a_1, a_2)\, N^2(a_3, K+a_4+\dots+a_{2n})]$$
$$\times [S^2(a_3, K+a_4+\dots+a_{2n}) N^2(-a_1, a_2) - S^2(a_1, a_2) N^2(a_3, K+a_4+\dots+a_{2n})]$$
$$\div [S^2(a_3, K+a_4+\dots+a_{2n})\, k^2 \operatorname{sn}^2 a_2 - N^2(a_3, K+a_4+\dots+a_{2n})];$$

and, from what we have shown before, we see that, to a factor *près*

(which factor does not contain a_1),

$$\mathrm{sn}^4 a_1 + C_6 \mathrm{sn}^2 a_1 + C_7 \mathrm{sn}\, a_1 \mathrm{cn}\, a_1 \mathrm{dn}\, a_1 + C_8$$
$$= S^2(a_3, K+a_4+\ldots+a_{2n}) N^2(a_1, a_2) - S^2(a_1, a_2) N^2(a_3, K+a_4+\ldots+a_{2n}).$$

And so we have

$$C(a_1, a_2, a_3, \ldots a_{2n}) C(-a_1, -a_2, a_3, \ldots a_{2n})$$
$$= D(a_2, a_3, \ldots a_{2n}) D(-a_2, a_3, \ldots a_{2n}) S(a_1, a_3) S(a_1, a_4) \ldots S(a_1, a_{2n})$$
$$\times \frac{S^2(a_3, K+a_4+\ldots+a_{2n}) N^2(a_1, a_2) - S^2(a_1, a_2) N^2(a_3, K+a_4+\ldots+a_{2n})}{k^2 \mathrm{sn}^2 a_2 S^2(a_3, K+a_4+\ldots+a_{2n}) - N^2(a_3, K+a_4+\ldots+a_{2n})}$$
$$= D(a_2, a_3, \ldots a_{2n}) D(-a_2, a_3, \ldots a_{2n}) S(a_1, a_3) S(a_1, a_4) \ldots S(a_1, a_{2n})$$
$$\times \frac{\mathrm{sn}^2(K+a_3+a_4+\ldots+a_{2n}) - \mathrm{sn}^2(a_1+a_2)}{k^2 \mathrm{sn}^2 a_2 \mathrm{sn}^2(K+a_3+a_4+\ldots+a_{2n}) - 1} \cdot N^2(a_1, a_2)$$
$$= \Theta_2(a_1+a_2+\ldots+a_{2n}) \Theta_2(-a_1-a_2+a_3+\ldots+a_{2n})$$
$$\times V(a_1, a_2, \ldots a_{2n}) V(-a_1, -a_2, a_3, \ldots a_{2n}),$$

proceeding precisely as before. So that we have, the sign being determined as before,

$$C(a_1, a_2, a_3, \ldots a_{2n}) = (-1)^{n-1} \Theta_2(a_1+a_2+\ldots+a_{2n}) V(a_1, a_2, \ldots a_{2n}).$$

And if we examine

$$D(a_1, a_2, \ldots a_{2n}) D(-a_1, -a_2, a_3, \ldots a_{2n}),$$

we see that it can be written in the form

$$A_5 + A_6 \mathrm{sn}\, a_1 \mathrm{cn}\, a_1 \mathrm{dn}\, a_1,$$

where A_5, A_6 only contain a_1 through even powers of $\mathrm{sn}\, a_1$. Also we have

$$D(-a_1, a_2, \ldots a_{2n}) D(a_1, -a_2, a_3, \ldots a_{2n})$$
$$= A_5 - A_6 \mathrm{sn}\, a_1 \mathrm{cn}\, a_1 \mathrm{dn}\, a_1.$$

Further, both A_5 and A_6 must evidently contain as factors

$$S(a_1, a_3) S(a_1, a_4) \ldots S(a_1, a_{2n}).$$

Hence it is clear that we may write

$$D(a_1, a_2, \ldots a_{2n}) D(-a_1, -a_2, a_3, \ldots a_{2n})$$
$$= k^2 C(a_2, a_3, \ldots a_{2n}) C(-a_2, a_3, \ldots a_{2n}) S(a_1, a_3) S(a_1, a_4) \ldots (a_1, a_{2n})$$
$$\times \{\mathrm{sn}^4 a_1 + C_{10} \mathrm{sn}^2 a_1 + C_{11} \mathrm{sn}\, a_1 \mathrm{cn}\, a_1 \mathrm{dn}\, a_1 + C_{12}\};$$

and

$$D(-a_1, a_2, \ldots a_{2n}) D(a_1, -a_2, a_3, \ldots a_{2n})$$
$$= k^2 C(a_2, a_3, \ldots a_{2n}) C(-a_2, a_3, \ldots a_{2n}) S(a_1, a_3) S(a_1, a_4) \ldots S(a_1, a_{2n})$$
$$\times \{\mathrm{sn}^4 a_1 + C_{10} \mathrm{sn}^2 a_1 - C_{11} \mathrm{sn}\, a_1 \mathrm{cn}\, a_1 \mathrm{dn}\, a_1 + C_{12}\}.$$

It follows from this that

$$\{\mathrm{sn}^4 a_1 + C_{10}\,\mathrm{sn}^2 a_1 + C_{11}\,\mathrm{sn}\,a_1\,\mathrm{cn}\,a_1\,\mathrm{dn}\,a_1 + C_{12}\}$$
$$\{\mathrm{sn}^4 a_1 + C_{10}\,\mathrm{sn}^2 a_1 - C_{11}\,\mathrm{sn}\,a_1\,\mathrm{cn}\,a_1\,\mathrm{dn}\,a_1 + C_{12}\}$$

$$= S^2(a_1, a_2)\left\{\mathrm{sn}^2 a_1 - \frac{D^2(a_2, a_3, \ldots a_{2n})}{k^2 C^2(a_2, a_3, \ldots a_{2n})}\right\} \times \left\{\mathrm{sn}^2 a_1 - \frac{D^2(-a_2, a_3, \ldots a_{2n})}{k^2 C^2(-a_2, a_3, \ldots a_{2n})}\right\}$$

$$= S^2(a_1, a_2)\, S(a_1, K+iK'+a_2+\ldots+a_{2n})\, S(a_1, K+iK'-a_2+a_3+\ldots+a_{2n})$$

$$= [S^2(a_3, K+iK'+a_4+\ldots+a_{2n})\, N^2(a_1, a_2) - S^2(a_1, a_2)\, N^2(a_3, K+iK'+a_4+\ldots+a_{2n})]$$

$$\times [S^2(a_3, K+iK'+a_4+\ldots+a_{2n})\, N^2(-a_1, a_2) - S^2(-a_1, a_2)\, N^2(a_3, K+iK'+a_4+\ldots+a_{2n})]$$

$$\div [S^2(a_3, K+iK'+a_4+\ldots+a_{2n})\, k^2\,\mathrm{sn}^2 a_2 - N^2(a_3, K+iK'+a_4+\ldots+a_{2n})];$$

and, from what we have shown before, we see that, to a factor *près* (which factor does not contain a_1),

$$\mathrm{sn}^4 a_1 + C_{10}\,\mathrm{sn}^2 a_1 + C_{11}\,\mathrm{sn}\,a_1\,\mathrm{cn}\,a_1\,\mathrm{dn}\,a_1 + C_{12}$$
$$= S^2(a_3, K+iK'+a_4+\ldots+a_{2n})\, N^2(a_1, a_2) - S^2(a_1, a_2)\, N^2(a_3, K+iK'+a_4+\ldots+a_{2n}).$$

And so we have

$$D(a_1, a_2, \ldots a_{2n})\, D(-a_1, -a_2, a_3, \ldots a_{2n})$$
$$= k^2 C(a_2, a_3, \ldots a_{2n})\, C(-a_2, a_3, \ldots a_{2n})\, S(a_1, a_3)\, S(a_1, a_4) \ldots S(a_1, a_{2n})$$
$$\times \frac{S^2(a_3, K+iK'+a_4+\ldots+a_{2n})\, N^2(a_1, a_2) - S^2(a_1, a_2)\, N^2(a_3, K+iK'+a_4+\ldots+a_{2n})}{k^2\,\mathrm{sn}^2 a_2\, S^2(a_3, K+iK'+a_4+\ldots+a_{2n}) - N^2(a_3, K+iK'+a_4+\ldots+a_{2n})}$$

$$= k^2 C(a_2, a_3, \ldots a_{2n})\, C(-a_2, a_3, \ldots a_{2n})\, S(a_1, a_3)\, S(a_1, a_4) \ldots S(a_1, a_{2n}) \times \frac{\mathrm{sn}^2(K+iK'+a_3+\ldots+a_{2n}) - \mathrm{sn}^2(a_1+a_2)}{k^2\,\mathrm{sn}^2 a_2\,\mathrm{sn}^2(K+iK'+a_3+\ldots+a_{2n}) - 1} \cdot N^2(a_1, a_2)$$

$$= C(a_2, a_3, \ldots a_{2n})\, C(-a_2, a_3, \ldots a_{2n})\, S(a_1, a_3)\, S(a_1, a_4) \ldots S(a_1, a_{2n}) \times \frac{\mathrm{cn}^2(a_3+\ldots+a_{2n}) - \mathrm{sn}^2(a_1+a_2)\,\mathrm{dn}^2(a_3+\ldots+a_{2n})}{\mathrm{sn}^2 a_2\,\mathrm{cn}^2(a_3+a_4+\ldots+a_{2n}) - \mathrm{dn}^2(a_3+a_4+\ldots+a_{2n})} \cdot N^2(a_1, a_2)$$

$$= C(a_2, a_3, \ldots a_{2n})\, C(-a_2, a_3, \ldots a_{2n})\, S(a_1, a_3)\, S(a_1, a_4) \ldots S(a_1, a_{2n}) \times \frac{1 - k^2\,\mathrm{sn}^2(a_1+a_2)\,\mathrm{sn}^2(K+a_3+\ldots+a_{2n})}{\mathrm{sn}^2 a_2 - \mathrm{sn}^2(K+a_3+\ldots+a_{2n})} \cdot N^2(a_1, a_2)$$

$$= \Theta_2(a_2+a_3+\ldots+a_{2n})\,\Theta_2(-a_2+a_3+\ldots+a_{2n}) \times V(a_2, a_3, \ldots a_{2n})\,V(-a_2, a_3, \ldots a_{2n})$$
$$\times \frac{\Theta^{4n-4}0\,\Theta_1(a_1-a_3)\,\Theta_1(a_1+a_3)\ldots\Theta_1(a_1-a_{2n})\,\Theta_1(a_1+a_{2n})}{\Theta^{4n-4}a_1\Theta^2a_3\Theta^2a_4\ldots\Theta^2a_{2n}}$$
$$\times \frac{\Theta^20\,\Theta_3(a_1+a_2+\ldots+a_{2n})\,\Theta_3(-a_1-a_2+a_3+\ldots+a_{2n})}{\Theta^2(a_1+a_2)\,\Theta_3^2(a_3+a_4+\ldots+a_{2n})}$$
$$\times \frac{\Theta^2a_2\Theta_3^2(a_3+a_4+\ldots+a_{2n})}{\Theta^20\,\Theta_2(a_2+a_3+\ldots+a_{2n})\,\Theta_2(-a_2+a_3+\ldots+a_{2n})}$$
$$\times \frac{\Theta^40\,\Theta_1(a_1-a_2)\,\Theta_1(a_2-a_1)}{\Theta^4a_1\Theta^4a_2}\,\Theta^2(a_1+a_2)$$

$$= V(a_2, a_3, \ldots a_{2n})\,V(-a_2, a_3, \ldots a_{2n})$$
$$\times \frac{\Theta^{4n}0\,\Theta_1(a_1-a_2)\Theta_1(a_2-a_1)\Theta_1(a_1+a_3)\Theta_1(a_1-a_3)\ldots\Theta_1(a_1-a_{2n})}{\Theta^{4n}a_1\Theta^2a_2\Theta^2a_3\ldots\Theta^2a_{2n}}\Theta_1(a_1+a_{2n})$$
$$\times\Theta_3(a_1+a_2+\ldots+a_{2n})\,\Theta_3(-a_1-a_2+a_3+\ldots+a_{2n})$$
$$= \Theta_3(a_1+a_2+a_3+\ldots+a_{2n})\,\Theta_3(-a_1-a_2+a_3+\ldots+a_{2n}) \times V(a_1, a_2, \ldots a_{2n})\,V(-a_1, -a_2, a_3, \ldots a_{2n}).$$

Whence, as before,

$$D(a_1, a_2, \ldots a_{2n}) = (-1)^{n-1}\,\Theta_3(a_1+a_2+\ldots+a_{2n})\,V(a_1, a_2, \ldots a_{2n}).$$

And, if we examine

$$N(a_1, a_2, \ldots a_{2n})\,N(-a_1, -a_2, \ldots a_{2n}),$$

we see that it can be written in the form

$$A_7+A_8\,\mathrm{sn}\,a_1\,\mathrm{cn}\,a_1\,\mathrm{dn}\,a_1,$$

where A_7, A_8 only contain a_1 through even powers of $\mathrm{sn}\,a_1$. Also, we have

$$N(-a_1, a_2, \ldots a_{2n})\,N(a_1, -a_2, \ldots a_{2n}) = A_7-A_8\,\mathrm{sn}\,a_1\,\mathrm{cn}\,a_1\,\mathrm{dn}\,a_1.$$

Further, both A_7 and A_8 must evidently contain as factors

$$S(a_1, a_3)\,S(a_1, a_4)\ldots S(a_1, a_{2n}).$$

Hence it is clear that we may write

$$N(a_1, a_2, \ldots a_{2n})\,N(-a_1, -a_2, \ldots a_{2n})$$
$$= k^2S(a_2, a_3, \ldots a_{2n})\,S(-a_2, a_3, \ldots a_{2n})\,S(a_1, a_3)\,S(a_1, a_4)\ldots S(a_1, a_{2n})$$
$$\times\{\mathrm{sn}^4a_1+C_{14}\,\mathrm{sn}^2a_1+C_{15}\,\mathrm{sn}\,a_1\,\mathrm{cn}\,a_1\,\mathrm{dn}\,a_1+C_{16}\};$$

and

$$N(-a_1, a_2, \ldots a_{2n})\,N(a_1, -a_2, \ldots a_{2n})$$
$$= k^2S(a_2, a_3, \ldots a_{2n})\,S(-a_2, a_3, \ldots a_{2n})\,S(a_1, a_3)\,S(a_1, a_4)\ldots S(a_1, a_{2n})$$
$$\times\{\mathrm{sn}^4a_1+C_{14}\,\mathrm{sn}^2a_1-C_{15}\,\mathrm{sn}\,a_1\,\mathrm{cn}\,a_1\,\mathrm{dn}\,a_1+C_{16}\}.$$

It follows from this that

$$\{\mathrm{sn}^4 a_1 + C_{14}\,\mathrm{sn}^2 a_1 + C_{15}\,\mathrm{sn}\,a_1\,\mathrm{cn}\,a_1\,\mathrm{dn}\,a_1 + C_{16}\}$$
$$\times \{\mathrm{sn}^4 a_1 + C_{14}\,\mathrm{sn}^2 a_1 - C_{15}\,\mathrm{sn}\,a_1\,\mathrm{cn}\,a_1\,\mathrm{dn}\,a_1 + C_{16}\}$$
$$= S^2(a_1, a_2)\left\{\mathrm{sn}^2 a_1 - \frac{N^2(a_2, a_3, \dots a_{2n})}{k^2 S^2(a_2, a_3, \dots a_{2n})}\right\}\left\{\mathrm{sn}^2 a_1 - \frac{N^2(-a_2, a_3, \dots a_{2n})}{k^2 S^2(-a_2, a_3, \dots a_{2n})}\right\}$$
$$= S^2(a_1, a_2)\, S(a_1, iK' + a_2 + \dots + a_{2n})\, S(a_1, iK' - a_2 + a_3 + \dots + a_{2n})$$
$$= [S^2(a_3, iK' + a_4 + \dots + a_{2n}) N^2(a_1, a_2) - S^2(a_1, a_2) N^2(a_3, iK' + a_4 + \dots + a_{2n})]$$
$$\times [S^2(a_3, iK' + a_4 + \dots + a_{2n})\, N^2(-a_1, a_2)$$
$$- S^2(-a_1, a_2)\, N^2(a_3, iK' + a_4 + \dots + a_{2n})]$$
$$\div [S^2(a_3, iK' + a_4 + \dots + a_{2n})\, k^2 \mathrm{sn}^2 a_2 - N^2(a_3, iK' + a_4 + \dots + a_{2n})];$$

and, from what we have shown before, we see that, to a factor *près* (which factor does not contain a_1),

$$\mathrm{sn}^4 a_1 + C_{14}\,\mathrm{sn}^2 a_1 + C_{15}\,\mathrm{sn}\,a_1\,\mathrm{cn}\,a_1\,\mathrm{dn}\,a_1 + C_{16}$$
$$= S^2(a_3, iK' + a_4 + \dots + a_{2n}) N^2(a_1, a_2) - S^2(a_1, a_2) N^2(a_3 + iK' + a_4 + \dots + a_{2n}).$$

And so we have

$$N(a_1, a_2, \dots a_{2n})\, N(-a_1, -a_2, \dots a_{2n})$$
$$= k^2 S(a_2, a_3, \dots a_{2n})\, S(-a_2, a_3, \dots a_{2n})\, S(a_1, a_3)\, S(a_1, a_4) \dots S(a_1, a_{2n})$$
$$\times \frac{S^2(a_3, iK' + a_4 + \dots + a_{2n}) N^2(a_1, a_2) - S^2(a_1, a_2) N^2(a_3, iK' + a_4 + \dots + a_{2n})}{k^2 \mathrm{sn}^2 a_2\, S^2(a_3, iK' + a_4 + \dots + a_{2n}) - N^2(a_3, iK' + a_4 + \dots + a_{2n})}$$
$$= k^2 S(a_2, a_3, \dots a_{2n})\, S(-a_2, a_3, \dots a_{2n})\, S(a_1, a_3)\, S(a_1, a_4) \dots (a_1, a_{2n})$$
$$\times \frac{\mathrm{sn}^2(iK' + a_3 + \dots + a_{2n}) - \mathrm{sn}^2(a_1 + a_2)}{k^2 \mathrm{sn}^2 a_2\, \mathrm{sn}^2(iK' + a_3 + \dots + a_{2n}) - 1}\, N^2(a_1, a_2)$$
$$= S(a_2, a_3, \dots a_{2n})\, S(-a_2, a_3, \dots a_{2n})\, S(a_1, a_3)\, S(a_1, a_4) \dots S(a_1, a_{2n})$$
$$\times \frac{1 - k^2 \mathrm{sn}^2(a_1 + a_2)\, \mathrm{sn}^2(a_3 + a_4 + \dots + a_{2n})}{\mathrm{sn}^2 a_2 - \mathrm{sn}^2(a_3 + a_4 + \dots + a_{2n})} \cdot N^2(a_1, a_2)$$
$$= \Theta_1(a_2 + a_3 + \dots + a_{2n})\, \Theta_1(-a_2 + a_3 + \dots + a_{2n})$$
$$\times V(a_2, a_3, \dots a_{2n})\, V(-a_2, a_3, \dots a_{2n})$$
$$\times \frac{\Theta^{4n-4} 0\, \Theta_1(a_1 - a_3)\, \Theta_1(a_1 + a_3) \dots \Theta_1(a_1 - a_{2n})\, \Theta_1(a_1 + a_{2n})}{\Theta^{4n-4} a_1 \Theta^2 a_3 \Theta^2 a_4 \dots \Theta^2 a_{2n}}$$
$$\times \frac{\Theta^2 0\, \Theta(a_1 + a_2 + \dots + a_{2n})\, \Theta(-a_1 - a_2 + a_3 + \dots + a_{2n})}{\Theta^2(a_1 + a_2)\, \Theta^2(a_3 + a_4 + \dots + a_{2n})}$$
$$\times \frac{\Theta^2 a_2\, \Theta^2(a_3 + a_4 + \dots + a_{2n})}{\Theta^2 0\, \Theta_1(a_2 + a_3 + \dots + a_{2n})\, \Theta_1(-a_2 + a_3 + \dots + a_{2n})}$$
$$\times \frac{\Theta^4 0\, \Theta_1(a_1 - a_2)\, \Theta_1(a_2 - a_1)}{\Theta^4 a_1 \Theta^4 a_2}\, \Theta^2(a_1 + a_2)$$

$$= V(a_2, a_3, \dots a_{2n})\, V(-a_2, a_3, \dots a_{2n})$$

$$\times \frac{\Theta^{4n}0\, \Theta_1(a_1-a_2)\, \Theta_1(a_2-a_1)\, \Theta_1(a_1-a_3)\, \Theta_1(a_1+a_3) \dots \dots \Theta_1(a_1-a_2)\, \Theta_1(a_1+a_{2n})}{\Theta^4 a_1 \Theta^2 a_2 \Theta^2 a_3 \dots \Theta^2 a_{2n}}$$

$$\times \Theta(a_1+a_2+\dots+a_{2n})\, \Theta(-a_1-a_2+a_3+\dots+a_{2n})$$

$$= \Theta(a_1+a_2+\dots+a_{2n})\, \Theta(-a_1-a_2+a_3+\dots+a_{2n})$$

$$\times V(a_1, a_2, \dots a_{2n})\, V(-a_1, -a_2, a_3, \dots a_{2n}).$$

And so we have, as before,

$$N(a_1, a_2, \dots a_{2n}) = (-1)^{n-1}\, \Theta(a_1+a_2+\dots+a_{2n})\, V(a_1, a_2, a_3, \dots a_{2n}).$$

So that, when the law holds for an odd number, it holds for the succeeding even number; and we have already shown that when it holds for an even number, it holds for the succeeding odd number; and we have the law for the number 2. We have thus an induction showing that the law is generally true for positive integers.

On the Coordinates of a Plane Curve in Space.

By H. W. Lloyd Tanner, M.A.

[*Read April 6th*, 1882.]

In the following is proposed a theory of the Coordinates of a Plane Curve in Space, analogous to the six coordinates of a straight line or the twenty-one coordinates of a conic in space. This theory is based upon the use of the symbolical equation to a surface which determines the curve; viz., an equation of the form

$$(ax+by+cz+dw)^n = 0.$$

The use of the equation for this purpose was suggested by a question (6998), proposed by Mr. W. R. Westropp Roberts in the *Educational Times* for February: and, the idea once suggested, the development was easy, especially as Mr. Spottiswoode's Paper "On the Twenty-one Coordinates of a Conic in Space," and Professor Cayley's addition thereto (*Proceedings of London Mathematical Society*, Vol. x., pp. 185—196) could be utilized, not merely to verify results obtained, but also as suggesting the course and methods of the research.

The outcome of the investigation is, that any plane curve is completely defined by a certain number of "coordinates." These coordinates are the products of n dimensions of six "umbræ," which are analogous to the coordinates of a right line. These, being linear homogeneous

functions of the a, b, c, d above, are purely symbolical, except when in products of n dimensions, such as the coordinates.

Of the coordinates there is a special group consisting of $2(n^2+3n-1)$ coordinates and including as a particular case the eighteen coordinates of a conic in space. This group seems to be sufficient to determine the curve completely. There is no doubt about this save in certain cases in which some of the equations employed are illusory. Even in these cases it seems probable that the "eighteen" group is sufficient to distinguish the curve, but I have not succeeded in proving that this is actually the case.

The equations obtained have for the most part, a striking likeness to that which expresses that two lines meet; viz., to

$$a_1f_2+f_1a_2+b_1g_2+g_1b_2+c_1h_2+h_1c_2=0.$$

Below is a brief statement of the contents, that may be convenient:—

§§ 1—5 Definitions, notation, and fundamental equations.
§§ 6—8 Identical relations between coordinates.
§ 9 Second process for deducing these.
§§ 10, 11 Fallacious method.
§§ 12 Group of mutually independent coordinates sufficient to define the curve.
§§ 13, 14 Algebraic treatment of same question.
§ 15 Determination of plane of curve.
§§ 16—19 Properties of curves, certain coordinates of which vanish.
§ 20 Are the $2(n^2+3n-1)$ coordinates sufficient to determine a curve?
§§ 21—28 Various conditions expressed in terms of coordinates.
§ 29 Concerning the coordinates of the curve of intersection of two surfaces.

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1. A plane curve of the n^{th} degree is determined by the intersection of a plane

$$\alpha x+\beta y+\gamma z+\delta w=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

with a surface of the n^{th} degree whose equation may be written

$$(ax+by+cz+dw)^n=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

For the purposes of this paper, the symbols a, b, c, d have no meaning except when they occur in such combinations as $a^pb^qc^rd^s$ where $p+q+r+s=n$. In other words, only those combinations of a, b, c, d are regarded as significant which occur in the expansion of the expression on the left of (2). As to these it is assumed that the coefficient of $x^py^qz^rw^s$ in (2) is equal to the corresponding coefficient in the general equation of a surface of the n^{th} degree: or, what is the

same thing, each combination $a^p b^q c^r d^s$ $(p+q+r+s=n)$ is an arbitrary number, and there is no relation whatever between such symbols with different values of p, q, r, s.

If $A, B, \ldots$ be linear homogeneous functions of a, b, c, d, with ordinary algebraic coefficients, then any product of $A, B, \ldots$ of n dimensions is an algebraic expression, for it involves the symbols a, b, c, d in interpretable combinations and in no other way.

2. From the two equations (1), (2), it is easy to eliminate each of the four variables x, y, z, w in turn. The first equation gives x, for example, in terms of y, z, w. Substituting in (2) and arranging, we get

$$\{(ab-\beta a)\,y-(\gamma a-ac)\,z+(ad-\delta a)\,w\}^n=0.$$

To simplify this and its fellows, put

$$\begin{aligned} F &= \beta c-\gamma b, & A &= ad-\delta a,\\ G &= \gamma a-ac, & B &= \beta d-\delta b,\\ H &= ab-\beta a, & C &= \gamma d-\delta c, \end{aligned}$$

or, as it may conveniently be written,

$$F, G, H, A, B, C = \begin{vmatrix} a, & \beta, & \gamma, & \delta \\ a, & b, & c, & d \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

The equation above obtained then becomes

$$(Hy-Gz+Aw)^n=0,$$

and the system of four equations formed by eliminating x, y, z, w in turn from (1), (2), may be written

$$\{(\quad . \quad H, \; -G, \; A \;\mathfrak{X}\; x, y, z, w)\}^n=0 \quad \ldots\ldots\ldots (4)$$
$$\begin{vmatrix} -H, & . & F, & B \\ G, & -F, & . & C \\ -A, & -B, & -C, & . \end{vmatrix}$$

Each of these equations represents a cone, the first having its vertex at the point $y=z=w=0$, and similarly for the others. The curve is completely defined by the intersection of the plane of the curve (1), with any one of the cones, the vertex of which does not lie in that plane; and, since the four vertices cannot lie in one plane, there is at least one of the cones which with (1) suffices to determine the curve. It will appear hereafter that, generally speaking, the plane (1) is uniquely determined by the system (4), so that this system gives the curve without ambiguity or irrelevant additions.

3. To form the equation to the cone passing through the curve (1, 2) and having its vertex at any point $(\xi, \eta, \zeta, \omega)$, we write, as usual, $x+\lambda\xi$, $y+\lambda\eta$, $z+\lambda\zeta$, $w+\lambda\omega$ for x, y, z, w in the equations (1), (2), and eliminate

λ. The equations from which λ is to be removed are

$$ax+\beta y+\gamma z+\delta w+\lambda(a\xi+\beta\eta+\gamma\zeta+\delta\omega)=0,$$
$$\{ax+by+cz+dw+\lambda(a\xi+b\eta+c\zeta+d\omega)\}^n=0.$$

The former determines λ as a scalar; and, by substituting this value in the latter, we get the result of the elimination in the form

$$\left\{\begin{vmatrix} ax+\beta y+\gamma z+\delta w, & a\xi+\beta\eta+\gamma\zeta+\delta\omega \\ ax+by+cz+dw, & a\xi+b\eta+c\zeta+d\omega \end{vmatrix}\right\}^n=0.$$

This may be written

$$\left\{\begin{vmatrix} a, & \delta \\ a, & d \end{vmatrix}\cdot\begin{vmatrix} \xi, & \omega \\ x, & w \end{vmatrix}+\begin{vmatrix} \beta, & \gamma \\ b, & c \end{vmatrix}\cdot\begin{vmatrix} \eta, & \zeta \\ y, & z \end{vmatrix}+\begin{vmatrix} \beta, & \delta \\ b, & d \end{vmatrix}\cdot\begin{vmatrix} \eta, & \omega \\ y, & w \end{vmatrix}\right.$$
$$\left.+\begin{vmatrix} \gamma, & a \\ c, & a \end{vmatrix}\cdot\begin{vmatrix} \zeta, & \xi \\ z, & x \end{vmatrix}+\begin{vmatrix} \gamma, & \delta \\ c, & d \end{vmatrix}\cdot\begin{vmatrix} \zeta, & \omega \\ z, & w \end{vmatrix}+\begin{vmatrix} a, & \beta \\ a, & b \end{vmatrix}\cdot\begin{vmatrix} \xi, & \eta \\ x, & y \end{vmatrix}\right\}^n=0.$$

If herein we replace the first factor of each term by its equivalent in (3), and the second factor from the system

$$\mathrm{a, b, c, f, g, h} = \begin{vmatrix} \xi, & \eta, & \zeta, & \omega \\ x, & y, & z, & w \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5),$$

the equation to the cone becomes

$$(Af+F\mathrm{a}+B\mathrm{g}+G\mathrm{b}+C\mathrm{h}+H\mathrm{c})^n=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(6).$$

This equation includes (4) as particular cases, as it ought. For instance, putting $\eta=\zeta=\omega=0$, we have

$$\mathrm{a}=\mathrm{g}=\mathrm{h}=0,$$
$$\mathrm{b, c, f,} = -\xi z, \xi y, \xi w.$$

The substitution of these values in (6) reproduces the first equation of the system (4), and the other three may be similarly obtained.

4. The quantities a, b, c, f, g, h are the coordinates of the straight line through $(\xi, \eta, \zeta, \omega)$ and (x, y, z, w) (Salmon's "Geometry of Three Dimensions," 3rd edition, Art. 51), and the equations to the line are any pair of

$$\begin{pmatrix} . & \mathrm{h}, & -\mathrm{g}, & \mathrm{a} \\ -\mathrm{h}, & . & \mathrm{f}, & \mathrm{b} \\ \mathrm{g}, & -\mathrm{f}, & . & \mathrm{c} \\ -\mathrm{a}, & -\mathrm{b}, & -\mathrm{c}, & . \end{pmatrix}(x', y', z', w')=0 \quad \ldots\ldots\ldots\ldots (7),$$

since these, in virtue of (5), are identically satisfied when

$$x', y', z', w' = x+\lambda\xi, y+\lambda\eta, z+\lambda\zeta, w+\lambda\omega.$$

It is obvious that they form a particular case of (4), viz., when $n=1$.

It may save some trouble to notice that the coordinates of a straight line as above defined, are not the same as those used by Mr. Spottiswoode

in his paper "On the Twenty-one Coordinates of a Conic in Space." (*Proceedings of London Mathematical Society*, Vol. x., p. 186.) In fact his a, b, c are the f, g, h above, and *vice versâ*. His notation of the conic-coordinates is based upon the same system. Nevertheless it seemed desirable, in the absence of any special reason to the contrary, to revert to the original notation, especially as Prof. Cayley in his addition to Mr. Spottiswoode's paper (*loc. cit.*, p. 196), has set the example.

5. It is convenient to have a special name for the six symbols A, B, C, F, G, H which are defined by (3) and appear in the equations (4), (6); we shall therefore refer to them as *umbræ*. For the case $n = 1$ the umbræ become the coordinates. The umbræ naturally divide into pairs A, F; B, G; C, H of *conjugate* umbræ.

Since the umbræ are linear homogeneous functions of a, b, c, d, any product of n dimensions in the umbræ is an algebraic expression involving the coefficients of the general equation of the surface (2) linearly, and the plane-coefficients α, β, γ, δ in the n^{th} degree. Such a product is called a *coordinate* of the curve. The number of these coordinates, being the number of all the products of n dimensions that can be formed from six letters, is

$$\frac{n+5\,!}{n\,!\;5\,!} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

Thus in the case of a conic ($n = 2$) there are 21 coordinates, which is right. All these coordinates appear in the equation (6).

Two coordinates are *conjugate* when each is formed from the other by the substitution

$$(AF)\,(BG)\,(CH).$$

A special group of the coordinates is formed by those which appear in the equations (4), and these are generally sufficient to characterize the curve. To find the number of coordinates in this group, observe that the number in each equation is

$$\tfrac{1}{2}(n+1)\,(n+2),$$

for this is the number of products of n dimensions that can be made with three letters. Each of the coordinates A^n, B^n, C^n, F^n, G^n, H^n appears twice in the system, but no other coordinate is repeated. Hence altogether there are

$$\begin{aligned} &2\,(n+1)\,(n+2)-6, \\ &\quad = 2\,(n^2+3n-1) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9) \end{aligned}$$

coordinates in the special group. This includes as a particular case the "eighteen coordinates" of a conic in space.

For the sake of reference, we give here the coordinates of a conic in terms of the six umbræ. The expressions on the left of the following equations are the conic-coordinates in Mr. Spottiswoode's notation (*Proceedings of London Mathematical Society*, Vol. x., p. 187); the letters on the right are the umbræ

$$\left.\begin{array}{rrrrr} F, & G, & H, & F', & G', \quad H' = A^2, B^2, C^2, BC, CA, AB \\ A, & B, & C, & -A', & -B', \quad -C' = F^2, G^2, H^2, GH, HF, FG \\ & & P, & -L', & L = AF, AG, AH \\ & & M, & Q, & -M' = BF, BG, BH \\ & & -N', & N, & R = CF, CG, CH \end{array}\right\} \ldots\ldots(10).$$

It will be noticed that the coordinates in the first row are conjugate to those in the second, each to each: while, in the square, the conjugates are symmetrically disposed around P, Q, R, each of which is self-conjugate. The inversion of the first and second rows is due to the circumstance noted at the end of § 4.

6. The identical relations subsisting between the coordinates may be derived from the identities

$$\begin{vmatrix} \alpha & \beta & \gamma & \delta \\ \alpha & \beta & \gamma & \delta \\ a & b & c & d \end{vmatrix} = 0$$

and

$$\begin{vmatrix} a & b & c & d \\ \alpha & \beta & \gamma & \delta \\ a & b & c & d \end{vmatrix} = 0.$$

The former gives the system

$$\begin{vmatrix} . & C, & -B, & F \\ -C, & . & A, & G \\ B, & -A, & . & H \\ -F, & -G, & -H, & . \end{vmatrix} (\alpha, \beta, \gamma, \delta) = 0 \ldots\ldots\ldots\ldots\ldots\ldots(11);$$

the latter leads to a similar system with a, b, c, d in the place of α, β γ, δ. The two systems are equivalent to the definitions (3): but it is unnecessary to take account of both sets, since we get the same results by eliminating α, β, γ, δ from (11), or a, b, c, d from the corresponding system.

The equations (11) are true, because, when expanded, the coefficients of a, b, c, d vanish identically. Clearly, then, they are true when multiplied by any quantity, P say; since the coefficients of aP, bP, &c. vanish. If we take P to be a product of umbræ of $n-1$ dimensions, we obtain a system of equations connecting α, β, γ, δ with the coordi-

nates of the curve. Hence, by the elimination of $\alpha, \beta, \gamma, \delta$, we get relations such as

$$\begin{vmatrix} . & CP_1, & -BP_1, & FP_1 \\ -CP_2, & . & AP_2, & GP_2 \\ BP_3, & -AP_3, & . & HP_3 \\ -FP_4, & -GP_4, & -HP_4, & . \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots (12),$$

where P_1, P_2, P_3, P_4 are products of $n-1$ umbræ. Other determinants may be formed by repeating the rows of the matrix in (11) with different multipliers. For instance, we have

$$\begin{vmatrix} CP_1, & BP_1, & FP_1 \\ CP_2, & BP_2, & FP_2 \\ CP_3, & BP_3, & FP_3 \end{vmatrix} = 0,$$

and others which it is not necessary to specify. It will be convenient at times to write C_1, B_1, &c., as abbreviations of CP_1, BP_1, &c.

7. Among the equations (12) there are some that are specially noteworthy. If P_1, P_2, P_3, P_4 are identical, the determinant is skew, and gives the identity

$$AP \,.\, FP + BP \,.\, GP + CP \,.\, HP = 0 \ldots\ldots\ldots\ldots\ldots (13).$$

If three of the multipliers be equal, the same result is obtained. When two multipliers are P and the other two are Q, we get an equation which, by the help of (13), reduces to

$$AP \,.\, FQ + FP \,.\, AQ + BP \,.\, GQ + GP \,.\, BQ + CP \,.\, HQ + HP \,.\, CQ = 0 \ldots (14),$$

or, say, $$A_1F_2 + F_1A_2 + B_1G_2 + G_1B_2 + C_1H_2 + H_1C_2 = 0.$$

The case in which two of the multipliers are equal, the other pair being unequal, is of some interest. When the determinant is developed, each term is of the fourth degree, but by the help of (13) it can be reduced to a form in which there is a factor common to every term. The remaining cubic expression takes a neat form when we make use of the notation

$$\tfrac{1}{2}(HAF) \equiv H_1A_2F_3 - A_1H_2F_3 - F_1A_2H_3 \ldots\ldots\ldots\ldots\ldots (15),$$

where H_1, A_2, &c. are written for the coordinates HP_1, AP_2, &c. The lopsided function $\frac{1}{2}(HAF)$ is, in fact, an unsymmetrical half of the determinant $(H_1A_2F_3)$, consisting of the positive term containing H_1, and the negative terms that contain A_1 and F_1 respectively. It is easy to verify that

$$\tfrac{1}{2}(HAF) - \tfrac{1}{2}(HFA) = (HAF).$$

Making use of this notation, the equation formed by supposing in (12) P_2 to be equal to P_1, may be written

$$\tfrac{1}{2}(HAF) + \tfrac{1}{2}(HBG) + \tfrac{1}{2}(HCH) = 0 \ldots\ldots\ldots\ldots\ldots (16).$$

Similarly, supposing $P_2 = P_3$, we get

$$\tfrac{1}{2}(FAF) + \tfrac{1}{2}(FBG) + \tfrac{1}{2}(FCH) = 0.$$

In fact, all the equations thus obtainable may be written down by putting A, B, C, F, G, H in turn in the first places of the symbols in (16).

8. There are also relations in which the coordinates are linearly involved.

Now, let the first equation of (11) be multiplied by AP, the second by BP, the third by CP, and the results added together; P being a product of $n-2$ umbræ. We thus get

$$(AFP + BGP + CHP)\,\delta = 0.$$

The process may be indicated thus :—

$$[AP]_\alpha + [BP]_\beta + [CP]_\gamma = 0, \; = (AFP + BGP + CHP)\,\delta.$$

Here the quantities in square brackets indicate the multipliers, and the subscripts show which equation of (11) is used, the subscript being that one of the four quantities $\alpha, \beta, \gamma, \delta$ which does not appear in the equation in question. This notation will be useful in the sequel.

The result shows that, unless $\delta = 0$, we have

$$AFP + BGP + CHP = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (17);$$

and herein, since P is of $n-2$ dimensions in the umbræ, each term is a coordinate of the curve. Similarly we get

$$[GP]_\alpha - [FP]_\beta + [CP]_\delta = 0, \; = -(AFP + BGP + CHP)\,\gamma,$$

so that (17) must be true unless $\gamma = 0$. And in the same way we may infer that, unless $\alpha, \beta, \gamma, \delta$ all vanish, (17) is true. As $\alpha, \beta, \gamma, \delta$ cannot all vanish, (17) is unconditionally true.

The equation (17) may also be derived from the equations (12). Put

$$P_1, P_2, P_3 = AP, BP, CP,$$

and, for clearness, write Q instead of P_4, which is left indeterminate. Then (12) becomes, after some reductions,

$$\{CAP \,.\, ABP \,.\, FQ + ABP \,.\, BCP \,.\, GQ + BCP \,.\, CAP \,.\, HQ\} \times \{AFP + BGP + CHP\} = 0 \quad \ldots\ldots\ldots\ldots\ldots (18).$$

And we may form three other equations with the same second factor. Each of these equations is a representative of

$$\frac{n+4!}{n-1!\,5!}$$

equations; for instance, in the one written above, Q may be any product of $n-1$ dimensions formed from the six umbræ. Now, if in any one of these equations the first factor does not vanish, the equation (17)

is proved. It will not be necessary to work out this, probably a laborious task. A comparison with the investigation given above indicates that the system

$$CAP \,.\, ABP \,.\, FQ + \&c. = 0,$$

corresponds to the assumption

$$\delta = 0.$$

This being so, it would follow that the vanishing of the first factors of all the equations like (18) implies that α, β, γ, δ all vanish, and therefore that all the coordinates are zero.

The equation (17) may be deduced far more easily from the identity

$$AF + BG + CH = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19),$$

by multiplying by any product of $n-2$ umbræ. But it was desirable to show that (17) was not a new relation, independent of those derivable from (12).

Since P in (17) is of $n-2$ dimensions, it follows that there are

$$\frac{n+3\,!}{n-2\,!\,5\,!}$$

linear equations represented in (17). In the case of the conic there is only one such equation, viz. (19); and, if we introduce Mr. Spottiswoode's notation (see § 4), we get

$$P + Q + R = 0,$$

which is his equation (8).

The formal likeness of the equations (13), (14), (16), (17) to the identity (19) is worthy of note.

9. There is yet another way in which the equations (11) may be treated. Hitherto each equation (11) has been multiplied by a factor such that the resulting equation is interpretable. We may, however, multiply by factors of less than $n-1$ dimensions in the umbræ, and so obtain equations which are not interpretable. From such equations it is permissible to eliminate α, β, γ, δ; for elimination only requires the operations of multiplication and addition. Now, the result of the elimination may be interpretable, although the several equations, or some of them, are not so. The question is, are any new identities thus obtained?

In the equation (12), suppose P_1, P_2, P_3, P_4 are such that the product $P_1P_2P_3P_4$ is of $n-4$ dimensions, $= Q$ suppose. Then (12) becomes

$$(AF + BG + CH)^2 Q = 0,$$

which, since Q is of $n-4$ dimensions in the umbræ, is linear in the coordinates. But this is a consequence of (17), as may be seen by writing AFQ, BGQ, CHQ in succession for P, and adding the results.

If, again, in (12), we leave P_4 of $n-1$ dimensions, but take P_1, P_2, P_3 such that their product is of $n-3$ dimensions, we obtain another consequence of (17). The coefficient of FP_4 in the result is

$$(AF+BG+CH)\,A\,P_1P_2P_3,$$

which vanishes by (17); and so for the coefficients of GP_4, HP_4.

Suppose, next, that we treat two of the equations (11) in a similar manner; for instance,

$$[P_1]_\alpha \equiv \qquad CP_1\beta - BP_1\gamma + FP_1\delta = 0,$$
$$[P_2]_\beta \equiv -CP_2\alpha \qquad + AP_2\gamma + GP_2\delta = 0,$$

where the product $P_1 . P_2$ is of $n-2$ dimensions. Eliminating γ between these, we get

$$BCP_1P_2\alpha - ACP_1P_2\beta - (AFP_1P_2 + BGP_1P_2)\,\delta = 0.$$

The coefficients of α, β, δ are of n dimensions, and when the last is reduced by means of (17), the resulting equation is

$$BCP_1P_2\alpha - ACP_1P_2\beta + HCP_1P_2\delta = 0,$$

or

$$[CP_1P_2]_\gamma = 0.$$

Similarly, by eliminating δ between the two symbolic equations, we should obtain

$$[CP_1P_2]_\delta = 0.$$

Hence the two symbolic equations

$$[P_1]_\alpha = 0, \quad [P_2]_\beta = 0,$$

are equivalent to the two algebraic equations

$$[CP_1P_2]_\gamma = 0, \quad [CP_1P_2]_\delta = 0.$$

The only case still remaining is that in which we take one equation of (11), and multiply it by two factors whose united degrees amount to $n-2$; say, the equations thus obtained are

$$[P_1]_\alpha = 0, \quad [P_2]_\alpha = 0.$$

From such a pair no result can be obtained. Neither of the quantities β, γ, δ can be eliminated separately, since every determinant such as

$$\begin{vmatrix} CP_1, & BP_1 \\ CP_2, & BP_2 \end{vmatrix}, = CP_1BP_2 - BP_1CP_2,$$

vanishes identically, the two terms in its expansion representing one and the same coordinate.

Thus it appears that all the relations obtainable by the process of this article, are included in those previously derived from (11).

10. Allusion has already been made to the second set of identities

$$\begin{vmatrix} . & C, & -B, & F \\ -C, & . & A, & G \\ B, & -A, & . & H \\ -F, & -G, & -H, & . \end{vmatrix} (a, b, c, d) = 0 \quad \ldots\ldots\ldots\ldots\ldots (20).$$

By the elimination of a, b, c, d between the equations here implied, we can only obtain the relations already derived by eliminating α, β, γ, δ from the corresponding system (11). There is, however, a somewhat seductive train of reasoning which leads to conclusions not otherwise obtained.

The first equation of (20) raised to the n^{th} power is

$$(Cb - Bc + Fd)^n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (21).$$

Comparing this with (2), it appears that the ideal point whose coordinates are $(0, C, -B, F)$ lies upon the surface (2). From the first equation of (11) it follows that the same point lies in the plane (1). That is, it is a point upon the curve. Hence, substituting in the first equation of (4), $(0, C, -B, F)$ for x, y, z, w, we have

$$(H \,.\, C + G \,.\, B + A \,.\, F)^n = 0.$$

The substitution may be made after the equation (4) is expanded, and the development of the expression just written consists of terms each of which is a product of two conjugate coordinates, one formed from the umbræ (H, G, A), the other from (C, B, F).

The equation just found, and others similar to it, are not included in or implied by the relations of §§ 6, 7, 8 above. And a trial for the case of the conic shows that they are not true. It is worth while to point out the fallacy in the argument, as it is closely connected with some matters that arise in the sequel.

11. The quantities a^n, $a^p b^{n-p}$ are real quantities, and their meaning is determined without ambiguity by the definitions of § 1. So also the symbols a^{2n}, $a^{2n-1} b$ are real and free from ambiguity; for the former is expressible as the product $a^n \,.\, a^n$, the latter as $a^n \,.\, a^{n-1} b$, and in no other way. But a symbol such as $a^{2n-2} b^2$ is not interpretable, without independent information. It may represent the product $a^n \,.\, a^{n-2} b^2$ or $a^{n-1} b \,.\, a^{n-1} b$; or, more generally, we may have

$$(\lambda + \mu)\, a^{2n-2} b^2 = \lambda a^n \,.\, a^{n-2} b + \mu a^{n-1} b \,.\, a^{n-1} b,$$

and the ratio $\lambda : \mu$ must be determined in each case by considering the manner in which the $a^{2n-2} b^2$ was obtained. For other cases a still greater complication arises. In the instances in which such expressions occur hereinafter, it is easy to interpret them by comparison with the results of known algebraical methods; and these cases seem to indicate the existence of comparatively simple rules of interpretation.

The application of these remarks to the subject of § 10 is obvious. The equation (21) is true because, when C, B, F are replaced by their values in (3), every power and product of b, c, d has a vanishing coefficient. The fallacy lies in the following step, in which it is tacitly assumed that in the expansion of (21) the two real factors of each term are formed in a particular manner, viz., one from (C, B, F) only, the other from (b, c, d) only; or, say, that (21) is equivalent to

$$(C.b-B.c+F.d)^n = 0.$$

This is an unjustified—and, as it appears, an unjustifiable—assumption.

The results previously obtained are not affected by this source of error, since in every case the symbolical expressions have been changed into their algebraical equivalents when the transformation was free from ambiguity.

12. In discussing the question as to how many of the coordinates are left independent after taking account of the relations already established, it is not convenient to follow the lead given by Mr. Spottiswoode for the case of a conic. The process adopted then was to show that the eighteen coordinates were connected by nine equations numbered (7), (*Proc. Lond. Math. Soc.*, Vol. x., p. 188), and that the remaining coordinates were expressible in terms of the eighteen.

In the general case there are $2(n^2+3n-1)$ coordinates of the special group, viz., all the coordinates $(A, G, H)^n$, $(F, B, H)^n$, $(F, G, C)^n$ and $(A, B, C)^n$. The relations between these coordinates directly deducible from (11) are not the only ones. Take, for example, the first equation of (11) (multiplied by a product of $n-1$ umbræ to make it interpretable). This is

$$[P_1]_a = CP_1\beta - BP_1\gamma + FP_1\delta = 0.$$

Now, if CP_1, BP_1, FP_1 are coordinates of the groups specified above, we must have $P_1 = C^{n-1}$, B^{n-1}, or F^{n-1},

so that there are only *three* equations. And we may in like manner infer that, from the four equations (11), we can only derive twelve equations, composed exclusively of the special coordinates. Eliminating the three ratios $\alpha : \beta : \gamma : \delta$, we have nine relations between these coordinates. Since this number is insufficient, except when $n = 2$, it follows that other relations must be found by eliminating from the equations implied by (11), not merely the ratios $\alpha : \beta : \gamma : \delta$, but also the coordinates not included in the special group.

It seems preferable to substitute for this process one which is suggested by geometrical considerations. Suppose for the present that α, β, γ, δ are known, so that the plane of the curve is given. The curve is then determined by at least one of the cones whose equations

are given in (4). To avoid needless generality, we assume this cone to be the last, viz.,

$$(Ax+By+Cz)^n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22).$$

The vertex of this cone being the point $x = y = z = 0$, it is assumed hereby that this point does not lie in the plane (1) of the curve—in other words, that δ does not vanish. Should $\delta = 0$, we have only to transform our equations by one of the substitutions

$$\left.\begin{array}{l} (G, C)\,(H, -B)\,(\alpha, \delta)\,(a, d)\,(x, w) \\ (H, A)\,(F, -C)\,(\beta, \delta)\,(b, d)\,(y, w) \\ (F, B)\,(G, -A)\,(\gamma, \delta)\,(c, d)\,(z, w) \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (23).$$

These leave the defining equations (1), (2), (3) unchanged; but they transform (22) into the first, second, or third respectively of (4), and effect the corresponding changes required in the concomitant equations.

Supposing, then, that $\alpha, \beta, \gamma, \delta$, and all the coordinates $(A, B, C)^n$ are given, it is easy to determine the equation to a cone through the curve, having its vertex at any given point. Comparing with (6), we are enabled to evaluate all the coordinates. There is no restriction upon the cone (22). Hence we learn that the coordinates $(A, B, C)^n$ are mutually independent, and that in terms of these and $\alpha, \beta, \gamma, \delta$ every other coordinate may be expressed.

13. The same conclusion may be proved algebraically. We have

$$[P_3]_\gamma \equiv BP_3\alpha - AP_3\beta + HP_3\delta = 0.$$

Putting herein

$$P_3 = (A, B, C)^{n-1},$$

and remembering that δ does not vanish, we obtain all the coordinates

$$H(A, B, C)^{n-1},$$

since the first two terms contain only known quantities. Then, putting

$$P_3 = H(A, B, C)^{n-2},$$

we obtain, in like manner, all the coordinates

$$H^2(A, B, C)^{n-2},$$

and, by continuing the process, we evidently can determine all the coordinates

$$(H, A, B, C)^n.$$

To effect this we have used

$$\frac{n+2\,!}{n-1\,!\,3!} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24)$$

equations, since P_3 has had all the forms

$$(H, A, B, C)^{n-1}.$$

If, now, in the equation

$$[P_2]_\beta \equiv -CP_2\alpha + BP_2\gamma + GP_2\delta = 0,$$

we write $$P_2 = (H, A, B, C)^{n-1},$$

the first two terms are known. Hence all coordinates

$$G\,(H, A, B, C)^{n-1}$$

are found. By continuing the process, we obtain all the coordinates

$$(G, H, A, B, C)^n,$$

and to do this we use $$\frac{n+3\,!}{n-1\,!\,4\,!} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(25)$$

equations, since P_2 is any one of the forms

$$(G, H, A, B, C)^{n-1}.$$

Lastly, by using the equations

$$[P_1]_\alpha \equiv CP_1\beta - BP_1\gamma + FP_1\delta = 0,$$

and ascribing to P_1 the $$\frac{n+4\,!}{n-1\,!\,5\,!} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(26)$$

forms $$(F, G, H, A, B, C)^{n-1},$$

we can obtain all the remaining coordinates.

In this process there has been no waste of power, each equation has determined a new coordinate. For the number of originally known coordinates $$(A, B, C)^n$$

$$= \frac{n+2\,!}{n\,!\,2\,!}.$$

Adding this to the number of equations used, we get

$$\frac{n+2\,!}{n\,!\,2\,!} + \frac{n+2\,!}{n-1\,!\,3\,!} + \frac{n+3\,!}{n-1\,!\,4\,!} + \frac{n+4\,!}{n-1\,!\,5\,!} = \frac{n+5\,!}{n\,!\,5\,!},$$

and this is the number of all the coordinates.

14. That the equations unused in the last article are merely consequences of those that were used, may be proved by means of the identities

$$\left.\begin{aligned} \delta\,[GP]_\gamma &= \delta\,[HP]_\beta - \alpha\,[AP]_\alpha - \beta\,[AP]_\beta - \gamma\,[AP]_\gamma \\ \delta\,[FP]_\gamma &= \delta\,[HP]_\alpha + \alpha\,[BP]_\alpha + \beta\,[BP]_\beta + \gamma\,[BP]_\gamma \\ \delta\,[FP]_\beta &= \delta\,[GP]_\alpha - \alpha\,[CP]_\alpha - \beta\,[CP]_\beta - \gamma\,[CP]_\gamma \\ \delta\,[P]_\delta &= -\alpha\,[P]_\alpha - \beta\,[P]_\beta - \gamma\,[P]_\gamma \end{aligned}\right\} \quad \ldots\ldots(27),$$

which are easily verified. The first of these equations shows that the equation $$[GP]_\gamma = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(28)$$

is a consequence of the equations

$$[HP]_\beta = 0, \quad [AP]_\alpha = 0, \quad [AP]_\beta = 0, \quad [AP]_\gamma = 0,$$

since δ by hypothesis does not vanish. If we give to P such a value

that the four equations last written are amongst those used in § 13, that is, if we put $P = (H, A, B, C)^{n-1}$,

(28) represents a set of equations not used in that paragraph. And, continuing the process step by step by means of the first three identities in (27), we prove that all the equations

$$[P]_\alpha = 0, \quad [P]_\beta = 0, \quad [P]_\gamma = 0,$$

are consequences of those used in § 13. The last identity (27) then shows that

$$[P]_\delta = 0$$

is a necessary result of the same system. That is, after determining all the coordinates in terms of α, β, γ, δ and $(A, B, C)^n$, there are no independent equations left to give any identical relations between these quantities.

15. It now remains to consider the equations by means of which α, β, γ, δ may be determined. In doing so we may omit the equations

$$[P]_\delta = 0,$$

since these are, by the last identity in (27), merely consequences of the equations $\quad [P]_\alpha = 0, \quad [P]_\beta = 0, \quad [P]_\gamma = 0.$

If we solve three equations

$$[P_1]_\alpha = 0, \quad [P_2]_\beta = 0, \quad [P_3]_\gamma = 0,$$

we get
$$\alpha : -\beta : \gamma : -\delta = \begin{vmatrix} . & CP_1, & -BP_1, & FP_1 \\ -CP_2, & . & AP_2, & GP_2 \\ BP_3, & -AP_3, & . & HP_3 \end{vmatrix} \quad \ldots\ldots\ldots(29).$$

Or, we may take three equations, such as

$$[P_1]_\alpha = 0, \quad [P_2]_\alpha = 0, \quad [P_3]_\beta = 0,$$

which lead to

$$\alpha : -\beta : \gamma : -\delta = \begin{vmatrix} . & CP_1, & -BP_1, & FP_1 \\ . & CP_2, & -BP_2, & FP_2 \\ -CP_3, & . & AP_3, & GP_3 \end{vmatrix} \quad \ldots\ldots\ldots(30).$$

The results may be somewhat simplified by making two of the multipliers equal. We cannot make all equal, even in (29), for the solution then becomes illusory, as might be expected from the identity

$$AP_1 \cdot [P_1]_\alpha + BP_1 \cdot [P_1]_\beta + CP_1 \cdot [P_1]_\gamma = 0 \ldots\ldots\ldots(31).$$

If we put the $P_2 = P_1$ in (29), that is, if we eliminate from

$$[P_1]_\alpha = 0, \quad [P_1]_\beta = 0, \quad [P_2]_\gamma = 0 \ldots\ldots\ldots\ldots\ldots(32),$$

the result is

$$\left.\begin{aligned} \alpha : \beta : \gamma : \delta = \; & AP_1 . HP_2 - AP_2 . HP_1 \\ & : BP_1 . HP_2 - BP_2 . HP_1 \\ & : FP_1 . AP_2 + GP_1 . BP_2 + CP_1 . HP_2 \\ & : AP_2 . BP_1 - AP_1 . BP_2 \end{aligned}\right\} \quad \ldots\ldots(33).$$

The same expression will be found by using the three equations

$$[P_1]_\beta = 0, \quad [P_1]_\gamma = 0, \quad [P_2]_\gamma = 0 \ldots\ldots\ldots\ldots\ldots\ldots(34),$$

which lead to a solution of the type (30). In fact, (34) differs from (32) only by having $[P_1]_\gamma$ in place of $[P_1]_\alpha$, and by (31) the change is indifferent since in each case we are given that $[P_1]_\beta$ vanishes. In these results from (32), (34), it is assumed that the coordinate CP_1, which has been removed by division, does not vanish.

No difficulty arises with these formulæ, save in certain exceptional cases in which groups of the coordinates vanish. It will be convenient to consider briefly the meaning of groups of zero-coordinates, before continuing the discussion of (29), (30), (33).

16. When all the coordinates containing A vanish—a statement which may conveniently be abbreviated into "when A vanishes,"—we get, from (4), (11), the reduced equations

$$(Hy - Gz)^n = 0, \quad (By + Cz)^n = 0 \ldots\ldots\ldots\ldots\ldots\ldots(35),$$

$$Ca - G\delta = 0, \qquad Ba + H\delta = 0 \ldots\ldots\ldots\ldots\ldots\ldots(36).$$

For the moment we exclude the case in which a second umbra vanishes. Then either of the equations (35) represents a series of planes through the line $y = 0$, $z = 0$, and the curve becomes a number of straight lines passing through the same point, viz., the point where the line $y = 0$, $z = 0$ meets the plane of the curve. But, if the plane of the curve contains the line $y = z = 0$, the equations (35) are insufficient to determine the curve. In this case, supposing the equations (35) to include p planes coinciding with the plane of the curve, the curve is made up of a curve of the p^{th} order (not necessarily a proper curve), and the straight line $y = z = 0$ counted $n - p$ times.

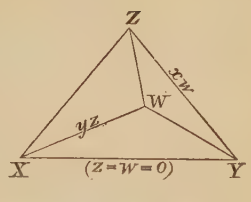
Similar conclusions are obtained by the consideration of (36). These equations are satisfied either when

$$\alpha = \delta = 0,$$

so that the curve-plane contains the line $y = z = 0$, or by having

$$\frac{GP_1}{CP_1} = \frac{GP_2}{CP_2} = \ldots = -\frac{HP_1}{BP_1} = -\frac{HP_2}{BP_2} = \ldots = \frac{\alpha}{\delta},$$

and this system expresses that the planes given by one equation of (35) are the same as those given by the other.



17. If two conjugate umbræ vanish, say A and F, three cases arise. It is clear that the plane of the curve cannot pass through both of the non-intersecting lines yz, xw. If it pass through the former, the curve consists of a pencil of straight lines passing through the point xw in the plane. And, *mutatis mutandi s* this holds good when the curve-plane passes through the line xw. If the plane of the curve contains neither of the lines yz, xw, the curve reduces to the line joining the points where yz, xw meet the plane. In this case it is geometrically evident that the plane of the curve is indeterminate.

If two non-conjugate umbræ, say A and H, both vanish, the curve must either lie in the plane $z = 0$, in which case its form is only restricted by the coordinates $(B, C, F, G)^n$; or it consists of a pencil of coplanar rays passing through the vertex $X\,(y = z = w = 0)$. The latter alternative is most simply deduced from the result of § 19 below, by observing that, if z does not vanish, the coordinate G^n must be zero, so that each of the coordinates $(A, G, H)^n$ vanishes. Or the two cases may be discussed by an analysis of the equations (4), (11), omitting the terms involving A and H.

18. When three umbræ vanish, three cases arise. First, two of the umbræ may be conjugate. In this case the curve can only consist of one of the edges of the tetrahedron of reference. Next, the three vanishing umbræ may be one of the groups (A, G, H), (F, B, H), (F, G, C) or (A, B, C), which appear in (4). In this case the curve lies in the plane $x = 0$, $y = 0$, $z = 0$, or $w = 0$, respectively. Thirdly, the umbræ in question may be (F, B, C), (A, G, C), (A, B, H), or (F, G, H), a grouping suggested by (11). The information hence derived is curiously small. Taking the first set, (F, B, C), (4) reduces to

$$(Hy - Gz + Aw)^n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (37),$$

$$H^n \cdot x^n = 0, \quad G^n \cdot x^n = 0, \quad A^n \cdot x^n = 0.$$

Hence, either x vanishes, or the three coordinates H^n, G^n, and A^n vanish. That is, either the curve lies in the plane $x = 0$, or, since in the equation (37) y^n, z^n, w^n do not appear, the cone with vertex X must pass through the vertices Y, Z, W, and the curve must pass through three fixed points, viz., the points where its plane is met by the lines XY, XZ, XW (*i.e.*, $z = w = 0$, $y = w = 0$, $y = z = 0$).

19. We will suppose now that the $\frac{1}{2}(n+1)(n+2)$ coordinates given by one of the forms

$$(A, G, H)^n, \quad (F, B, H)^n, \quad (F, G, C)^n, \quad \text{or} \quad (A, B, C)^n$$

vanish. This is, of course, a far less sweeping assumption than that three, or even one, of the umbræ should vanish. The discussion in § 13 shows that, if all the coordinates

$$(A, B, C)^n = 0,$$

then either $\delta = 0$, or all the coordinates are evanescent, and the latter alternative may safely be disregarded. Similarly, it may be inferred that the equations

$$(A, G, H)^n = 0, \quad (F, B, H)^n = 0, \quad (F, G, C)^n = 0$$

imply respectively that

$$\alpha = 0, \quad \beta = 0, \quad \gamma = 0.$$

Or these results may be deduced by means of the substitutions (23).

I have failed to deduce any property of the curve whose coordinates such as $(F, B, C)^n$ all vanish.

20. The question—Are the coordinates $(A, G, H)^n$, $(F, B, H)^n$, $(F, G, C)^n$, and $(A, B, C)^n$, which appear in the system (4), sufficient to determine the plane of the curve?—may now be discussed.

The most obvious way of attacking the problem is to utilize the equations (11); but the result is a failure. The equations are

$$\begin{aligned}
[P_1]_\alpha &\equiv \quad . \quad CP_1 . \beta - BP_1 . \gamma + FP_1 . \delta = 0, \\
[P_2]_\beta &\equiv -CP_2 . \alpha \quad . \quad + AP_2 . \gamma + GP_2 . \delta = 0, \\
[P_3]_\gamma &\equiv \quad BP_3 . \alpha - AP_3 . \beta \quad . \quad + HP_3 . \delta = 0, \\
[P_4]_\delta &\equiv -FP_4 . \alpha - GP_4 . \beta - HP_4 . \gamma \quad . \quad = 0.
\end{aligned}$$

If the coordinates herein involved are to be only those that occur in the selection mentioned above, then we must have

$$P_1 = C^{n-1}, \quad B^{n-1}, \quad \text{or} \quad F^{n-1},$$

and a corresponding choice for P_2, P_3, P_4.

But it is quite possible that the solutions obtained from these 12 equations may be illusory. This is the case, for instance, if all the coordinates that occur in the equations vanish; as, for example, in the case of a quartic curve lying in the plane $w = 0$, and having three nodes, one at each of the vertices X, Y, Z.

It must not, however, be inferred from this failure that the group of coordinates $(A, G, H)^n$, &c. are insufficient to determine the plane of the curve. It is quite possible that, given these coordinates, we may be able to determine others lying outside the group which will in turn fix the values of α, β, γ, δ. The analogous case discussed in § 12 tends to show the probability of this, and in the case of the particular curve quoted above it is easily shown to be true. Of that curve all the coordinates $(A, G, H)^n$ vanish, therefore by § 19 $\alpha = 0$; the coordi-

nates $(F, B, H)^n$, $(F, G, C)^n$ also vanish, therefore $\beta = 0$, $\gamma = 0$. So that by these coordinates the plane of the curve is, in fact, uniquely determined. But I have not succeeded in showing generally that, given all the coordinates of the "eighteen" group, the plane is uniquely determinable.

21. Since a curve is completely determined by its coordinates as above defined, any measurable property of the curve is expressible by means of its coordinates, and the conditions of any geometrical relation between a curve and other curves, points, or surfaces, may be similarly expressed. Some of these have already been given. For instance, the equations (4) are the conditions that a given curve should pass through a given point. The system (11) may be regarded as the conditions that a given curve should lie in a given plane. The condition that a curve should intersect a straight line, both being given by their coordinates, is the equation (6),

$$(Af + Fa + Bg + Gb + Ch + Hc)^n = 0 \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

and this includes Prof. Cayley's equation (*Proc. Lond. Math. Soc.*, Vol. x., p. 196), the notation being changed by means of the scheme (10).

Before proceeding to other relations, it will be convenient to mention some results in which straight lines only are concerned.

22. The condition that two lines should intersect is

$$a_1 f_2 + f_1 a_2 + b_1 g_2 + g_1 b_2 + c_1 h_2 + h_1 c_2 = 0,$$

which is indeed a particular case of (6). The same equation expresses that the two lines are coplanar. In fact, to say two lines meet is only another way of saying that they lie in the same plane. When, however, the question is about more than two lines, or about curves, the statements are not equivalent.

Consider the three lines whose coordinates are $(a_1, \ldots f_1, \ldots)$, $(a_2, \ldots f_2, \ldots)$, $(a_3, \ldots f_3, \ldots)$. The equations

$$\left.\begin{aligned} a_2 f_3 + f_2 a_3 + b_2 g_3 + g_2 b_3 + c_2 h_3 + h_2 c_3 &= 0 \\ a_3 f_1 + \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots &= 0 \\ a_1 f_2 + \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (38)$$

express that the three lines either meet in a point or lie in a plane. To distinguish the two cases, we may first seek the condition for intersection at one point by eliminating x, y, z, w from the equations to the three lines. It is easy to show, however, that the extra conditions are given by

$$(a_1 g_2 h_3) = 0, \ (f_1 b_2 h_3) = 0, \ (f_1 g_2 c_3) = 0, \ (a_1 b_2 c_3) = 0 \ldots\ldots (39),$$

where $(a_1g_2h_3)$, &c. are determinants, and generally speaking any one of these equations is sufficient. In fact, the equation to the plane containing the first line and the vertex W is

$$a_1x+b_1y+c_1z=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(40).$$

Hence the equation $$(a_1b_2c_3)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(41)$$

expresses that the planes through W, containing the three lines, intersect along a common line; or that from W a line can be drawn to meet the three given lines. Since, however, the lines meet two by two, it follows that the three points of intersection must coincide, or two of the planes (40) must be identical. In the latter case, we should have one of the sets of equations

$$a_1 : b_1 : c_1 = a_2 : b_2 : c_2 = a_3 : b_3 : c_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(42),$$

and the conditions (41) are valueless.

If, however, all the equations (39) are satisfied, the lines must pass through a point, in despite of disqualifying equations such as (42). To take the worst case, let the plane through the lines 1, 2 contain three of the vertices X, Y, Z. Then the first three equations of (39) are unavailable. If the third line does not pass through W, the last equation of (39) is unimpeachable. If the third line does pass through W, the last equation is disqualified; but then, since the lines are not in one plane, we know by (38) that they must pass through one point.

The conditions that three lines should be coplanar are

$$(f_1b_2c_3)=0,\ (a_1g_2c_3)=0,\ (a_1b_2h_3)=0,\ (f_1g_2h_3)=0\ldots\ldots(43),$$

in addition to (38), and generally one of these is sufficient. Any algebraic proof of the conditions (39) may be changed into a proof of (43) by replacing each coordinate by its conjugate, and starting from (11) instead of (4).

If the conclusion is correct, the product of any of the determinants (39) by any one of those in (43) ought to vanish in virtue of (38). To show this it is only needful to examine the two typical cases of a product of corresponding determinants, and one of non-corresponding determinants. Of the first type we have

$$(a_1g_2h_3)\,.\,(f_1,\ b_2,\ c_3).$$

Multiplying these by the ordinary rule, the result is, by reason of (38), a skew determinant, which being cubic vanishes. A product of the second type is $$(a_1g_2h_3)\,.\,(a_1g_2c_3).$$

On development this is

$$= \{(a_2g_3)\,h_1+(a_3g_1)\,h_2+(a_1g_2)\,h_3\}\{(a_2g_3)\,c_1+(a_3g_1)\,c_2+(a_1g_2)\,c_3\}$$
$$= c_1h_1\,(a_2g_3)^2+\ldots+(c_2h_3+h_2c_3)\,(a_3g_1)\,(a_1g_2)+\ldots$$

Replacing herein c_1h_1, $c_2h_3+h_2c_3$, &c., from (38), and the identity

$$af+bg+ch=0,$$

this becomes after a slight reduction

$$= (f_1a_2g_3)\,(a_1a_2g_3)+(b_1a_2g_3)\,(a_1g_2g_3),$$

each term of which vanishes identically.

23. The conditions that two curves given by their coordinates should lie in the same plane are easily obtained. We have merely to eliminate the ratios $\alpha:\beta:\gamma:\delta$ between a double set of equations formed from (11); one set from the coordinates of each curve. In any particular example care must be taken that the three equations selected for the purpose for each curve are mutually independent. Three conditions are obtained, since from six equations we have to eliminate three ratios. Indicating the umbræ of the two curves by subscripts, we may write the results in forms such as

$$A_1P\,.\,F_2Q+F_1P\,.\,A_2Q+B_1P\,.\,G_2Q$$
$$+G_1P\,.\,B_2Q+C_1P\,.\,H_2Q+H_1P\,.\,C_2Q=0,$$

$$\begin{vmatrix} C_1P, & B_1P, & F_1P \\ C_1Q, & B_1Q, & F_1Q \\ C_2R, & B_2R, & F_2R \end{vmatrix} = 0,$$

or, yet again, in the form of (16). Herein each constituent of the determinant is a coordinate.

In the case in which one curve is a straight line, there are only two conditions to satisfy; since the plane of the line may be any one of a singly infinite series of planes. These two may consist of the equation (6), and an equation such as

$$\begin{vmatrix} CP, & BP, & FP \\ CQ & BQ, & FQ \\ c, & b, & f \end{vmatrix} = 0.$$

24. The condition that two curves $(A_1\ldots F_1\ldots)^m$ and $(A_2\ldots F_2\ldots)^n$ should intersect is most simply obtained when the coordinates of the line of intersection of their plane is given,—say, they are $(a,\ldots f\ldots)$. The first curve intersects this line in m points; the second in n points; and we propose to give the condition that one point should be common to the two sets. Upon the conditions that two or more points should be common we do not touch.

If x, y, z, w be the coordinates of any point common to the curves

and to the line, we have

$$(A_1x+B_1y+C_1z)^m = 0,$$
$$(A_2x+B_2y+C_2z)^n = 0,$$
$$ax + by + cz = 0.$$

Eliminating x, y, z from these, we have

$$\left(\begin{vmatrix} A_1, & B_1, & C_1 \\ A_2, & B_2, & C_2 \\ a, & b, & c \end{vmatrix}\right)^{mn} = 0,$$

which may be compared with the last of (39); and the analogues of the other three equations (39) may be similarly obtained. The expression is, in the absence of a canon of interpretation, of little value except as a formula of verification. In the case of $m = n = 2$ the problem has been worked out by Mr. Spottiswoode (*Proc. Lond. Math. Soc.*, Vol. x., p. 193), and a comparison has served to detect one slight error, viz., in the coefficient of $4f^2gh$, $(GG')(HH')$ should be $2(GG')(HH')$, and a similar correction is required in the coefficients of $4fg^2h$, $4fgh^2$, in the writing out of his equation. The process used above was suggested by his.

25. If, however, we do not employ the coordinates of the line of intersection of the two planes, we may use the equations of four cones, two belonging to each curve, to eliminate x, y, z, w. By various selections we produce results similar in form to those of §§ 6, 7. The simplest form is obtained by using four equations, such as

$$(\quad . \qquad H_1y-G_1z+A_1w)^m = 0,$$
$$(-H_1x \quad . \quad +F_1z+B_1w)^m = 0,$$
$$(\quad G_2x-F_2y \quad . \quad +C_2w)^n = 0,$$
$$(\quad A_2x+B_2y+C_2z \quad . \quad)^n = 0.$$

The result of the elimination takes the form

$$\{C_1H_2(A_1F_2+A_2F_1+B_1G_2+\ldots)\}^{m^2n^2} = 0,$$

which may be compared with the equations (38), (6).

26. Given two curves in the same plane, required the condition that one should form a portion of the other. To determine this we utilize the equation (6), expressing the condition that an arbitrary right line meets a curve. The coordinates of the one curve being $(A_1 \ldots F_1 \ldots)^m$, and the smaller curve $(A_2 \ldots F_2 \ldots)^n$, every line that meets the latter must meet the former. Hence, whenever

$$(A_2f+F_2a+B_2g+\ldots)^n = 0,$$

we must also have $(A_1f+F_1a+B_1g+\ldots)^m = 0.$

The required condition therefore is that

$$(A_1\mathrm{f}+F_1\mathrm{a}+\ldots)^m \equiv (A_2\mathrm{f}+F_2\mathrm{a}+\ldots)^n \cdot (A'\mathrm{f}+F'\mathrm{a}+\ldots)^{m-n} \ldots (44),$$

where a, b, c, f, g, h are only subject to the relation

$$\mathrm{af}+\mathrm{bg}+\mathrm{ch}=0.$$

When the two curves are not coplanar, the condition (44) cannot be satisfied for all values of a ... b ..., but it is satisfied by the coordinates of a doubly infinite set of lines. If among these a set pass through one point not in the plane of either curve, then the curve $(A_2 \ldots F_2 \ldots)^n$ is in perspective with a portion of $(A_1 \ldots)^m$. To complete the discussion, we should replace a, b, &c., by their values given in (5), ξ, η, ζ, ω being the coordinates of the cone-vertex. The results obtained are not of obvious interest.

28. The problem of the last article suggests another:—to find the equation to the surface generated by a right line meeting three plane director curves.

Let the directors and any one of the generators be $(A_1 \ldots)^m$, $(A_2 \ldots)^n$, $(A_3 \ldots)^p$, and $(\mathrm{a}\ldots)$. We have the equations

$$\begin{aligned}
(A_1\mathrm{f}+B_1\mathrm{g}+C_1\mathrm{h}+F_1\mathrm{a}+G_1\mathrm{b}+H_1\mathrm{c})^m &= 0,\\
(A_2\mathrm{f}+\ \ldots\ \ \ldots\ \ \ldots\ \ \ldots\ )^n &= 0,\\
(A_3\mathrm{f}+\ \ldots\ \ \ldots\ \ \ldots\ \ \ldots\ )^p &= 0.
\end{aligned}$$

Also, if the coordinates of any point in the generator be x, y, z, w,

$$\begin{aligned}
. \ -z\mathrm{g}+y\mathrm{h}+w\mathrm{a} \ . \ \ . &= 0,\\
z\mathrm{f} \ . \ -x\mathrm{h} \ . \ +w\mathrm{b} \ . &= 0,\\
-y\mathrm{f}+x\mathrm{h} \ . \ \ . \ \ . \ w\mathrm{c} &= 0.
\end{aligned}$$

The last equation is introduced instead of eliminating one of the coordinates (a, ...) by means of the identity

$$\mathrm{af}+\mathrm{bg}+\mathrm{ch}=0.$$

Eliminating the six coordinates (a, ...) from the six equations, we get

$$\left(\begin{vmatrix}
A_1, & B_1, & C_1, & F_1, & G_1, & H_1\\
A_2, & B_2, & C_2, & F_2, & G_2, & H_2\\
A_3, & B_3, & C_3, & F_3, & G_3, & H_3\\
. & -z, & y, & w, & . & .\\
z, & . & -x, & . & w, & .\\
-y, & x, & . & . & . & w
\end{vmatrix}\right)^{mnp} = 0.$$

On development the factor w^{mnp} divides out. It is irrelevant, and was introduced by the sixth equation. It is unnecessary to work this out, for the result is practically given in Salmon's "Geometry of Three

Dimensions," 3rd edition, p. 79. It may be noted that, when two of the director curves are right lines, the above equation is immediately interpretable.

29. It may be worth notice that the coordinate system of representation is applicable to the intersection of two surfaces. Suppose these

are $$(ax+by+cz+dw)^m = 0,$$

and $$(a'x+b'y+c'z+d'w)^n = 0.$$

Adopting Prof. Cayley's method, we form the equation to the cone passing through the intersection of these surfaces and having its vertex at ξ, η, ζ, ω. This equation is, when the same notation as before is utilized, $$(Af+Fa+Bg+Gb+Ch+Hc)^{mn} = 0.$$

In this case the difficulties of interpretation begin with the fundamental equation itself.

In Salmon's "Geometry of Three Dimensions," 3rd edition, Art. 217, a particular case ($m = n = 2$) is given, and the result given above is consistent therewith.

On Polygons circumscribed about a Cuspidal Cubic.

By Mr. R. A. Roberts, M.A.

[*Read April 6th*, 1882.]

I propose to consider some cases in which an infinite number of closed polygons can be circumscribed about a cuspidal cubic and inscribed in another curve. In all the cases which I shall consider, the curve circumscribing the polygon is unicursal.

The equation of a cuspidal cubic being reduced to the form $y^3 = x^2z$, we may take $1, \vartheta, \vartheta^3$ as the coordinates of a point on the curve, and the equation of the tangent at the point ϑ is then $2\vartheta^3x - 3\vartheta^2y + z = 0$. Taking two tangents, we have, for the point x, y, z of their intersection,

$$2x = 1,\ 3y = t+u,\ z = t^2u \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where $u = \vartheta_1+\vartheta_2$, and $t = -\dfrac{\vartheta_1\vartheta_2}{\vartheta_1+\vartheta_2}$ is the parameter of the third tangent drawn from x, y, z.

Suppose we have $u = \dfrac{f_m}{f_n}$, where f_m, f_n are rational functions of t of the m^{th} and n^{th} degrees respectively; then the locus of x, y, z is a unicursal curve whose degree is equal to the greater of the numbers $n+1$,

$m+2$. This locus will touch the cubic in as many points as the equation $2u-t=0$ has roots; for, substituting the values (1) in the equation of the curve, we obtain $(2u-t)^2(u+4t)=0$. The locus will, evidently, also touch the stationary tangent z in the point corresponding to $t=0$.

Let us take the equation

$$\phi(\vartheta_1,\vartheta_2)=a(\vartheta_1+\vartheta_2)^2+b\vartheta_1^2\vartheta_2^2+f\vartheta_1\vartheta_2+g(\vartheta_1+\vartheta_2)+h\vartheta_1\vartheta_2(\vartheta_1+\vartheta_2)=0,$$

or

$$u=\frac{ft-g}{a-ht+bt^2};$$

then the locus of the intersection of the tangents is a unicursal cubic, touching the stationary tangent and having triple contact with the curve. This cubic is not limited in any further manner, as we have four constants at our disposal, and a unicursal cubic is determined by eight conditions.

Now, the equation $\phi(\vartheta_1,\vartheta_2)=0$ can, as is well known, be written in the form $u_1\pm u_2=\alpha$ a constant, where u_1 is an elliptic function of the first kind, depending on ϑ_1, and u_2 a similar function depending on ϑ_2. For a polygon of n sides, then, we have, taking the negative sign, $u_1-u_2=\alpha$, $u_2-u_3=\alpha$, &c., $u_{n-1}-u_n=\alpha$; from which we see that the polygon will close of itself, if $\alpha=\frac{2K}{n}$, which determines a relation between the constants a, b, &c. (see Schloemilch's "Théorie des Intégrales et des Fonctions Elliptiques," translated by Graindorge, p. 126). It is, in fact, by means of these equations, that the problem of inscribing polygons in one conic, which shall be circumscribed to another, is solved.

Thus, when the conditions stated above are satisfied, and also a further relation between the constants, we can have an infinite number of polygons circumscribed about a cuspidal cubic and inscribed in a nodal cubic. I proceed to find the two parameters corresponding to the nodes of this cubic. We could find them by the equations, in Salmon's "Curves," Art. 216 (c); but they can be more readily obtained by the following considerations. A node of the locus will evidently arise from the coexistence of the equations $\phi(t_1,\vartheta)=0$, $\phi(t_2,\vartheta)=0$, where t_1, t_2, ϑ are the parameters of three tangents passing through a point. Considering, then, $\phi(t,\vartheta)=0$ as a quadratic in t whose roots are t_1, t_2, we have

$$t_1+t_2=-\frac{\{h\vartheta^2+(2a+f)\vartheta+g\}}{a+h\vartheta+b\vartheta^2},$$

$$t_1t_2=\frac{a\vartheta^2+b\vartheta}{a+h\vartheta+b\vartheta^2},$$

which, combined with $t_1t_2+\vartheta(t_1+t_2)=0$, the condition that the tan-

gents should pass through a point, gives $a+f+h\vartheta=0$. For a cusp $t_1=t_2$, and we have, then, either

$$a^2+af-gh=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

or
$$h^2(a^2+af-gh)-4b(a+f)^3+4fh^2(a+f)=0\ldots\ldots\ldots\ldots(3),$$

the condition (2) being satisfied when the cusp is on z, the stationary tangent, and (3) when the cusp is on the given curve. Now, the equation (2) is the condition for the coexistence of the equations $\phi(\vartheta_1,\vartheta_2)=\phi(\vartheta_2,\vartheta_3)=\phi(\vartheta_3,\vartheta_1)=0$, for an indefinite number of values of ϑ_1, ϑ_2, ϑ_3, in which case an infinite number of triangles can be circumscribed about the given curve and inscribed in the locus. Hence we infer that, if a cuspidal cubic V have triple contact with a cuspidal cubic U, and the cusp of V lie on the stationary tangent of U, then an infinite number of triangles can be circumscribed about U and inscribed in V. When the relation (3) is satisfied, and also a further relation between the constants, we can have an infinite number of polygons circumscribing a cuspidal cubic U and inscribed in a cuspidal cubic V; the two curves U and V having double contact with each other, and being such that the cusp of V lies on U, and the stationary tangent of U is touched by V.

There is another case in which an infinite number of triangles can be circumscribed about a cuspidal cubic and inscribed in another cuspidal cubic. Let us take three points, whose parameters are ϑ_1, ϑ_2, ϑ_3, on the cubic $U\equiv y^3-x^2z=0$, such that

$$\vartheta_1+\vartheta_2+\vartheta_3=0,$$

$$4(\vartheta_1\vartheta_2+\vartheta_2\vartheta_3+\vartheta_3\vartheta_1)^3+27k(\vartheta_1\vartheta_2\vartheta_3)^2=0,$$

then it will be found that the tangents at these points form a triangle inscribed in the cubic $V\equiv y^3-kx^2z=0$. It may be remarked that the points of contact of the sides of this triangle lie on a line which touches $ky^3-x^2z=0$, and that the tangents to V at its vertices pass through a point which lies on $y^3-k^2x^2z=0$.

If we take the more general relation

$$\phi(\vartheta_1,\vartheta_2)=a(\vartheta_1+\vartheta_2)^2+b\vartheta_1^2\vartheta_2^2+c$$
$$+h\vartheta_1\vartheta_2(\vartheta_1+\vartheta_2)+g(\vartheta_1+\vartheta_2)+f\vartheta_1\vartheta_2=0,$$

or
$$u^2(a-ht+bt^2)+u(g-ft)+c=0,$$

the locus of the intersection of the tangents is a unicursal quartic, as may be seen thus :—If α and β are the roots of the equation

$$(g-ft)^2-4c(a-ht+bt^2)=0;$$

then, assuming $t=\dfrac{\alpha+\beta\lambda^2}{1+\lambda^2}$, we get for u an expression of the form

$\frac{1}{p+q\lambda+r\lambda^2}$, and, substituting these values in (1), we have

$$2x = (1+\lambda^2)(p+q\lambda+r\lambda^2),$$
$$3y = (\alpha+\beta\lambda^2)(p+q\lambda+r\lambda^2)+(1+\lambda^2)^2,$$
$$z = (\alpha+\beta\lambda^2)^2;$$

from which we see that the locus is a unicursal quartic touching the curve four times and the stationary tangent twice.

Thus, exactly as before, when a relation between the constants is satisfied, we can have an infinite number of polygons circumscribed about a cuspidal cubic and inscribed in a unicursal quartic which touches the cubic four times and its stationary tangent twice.

To determine the parameters of the nodes of this quartic, we have

$$\phi(t_1, \vartheta) = 0, \quad \phi(t_2, \vartheta) = 0, \quad t_1t_2+\vartheta(t_1+t_2) = 0,$$

whence
$$t_1+t_2 = -\frac{\{h\vartheta^2+(2a+f)\vartheta+c\}}{b\vartheta^2+h\vartheta+a},$$
$$t_1t_2 = \frac{a\vartheta^2+g\vartheta+c}{b\vartheta^2+h\vartheta+a},$$

ϑ being determined by the equation $h\vartheta^3+(a+f)\vartheta-c = 0$. If we eliminate ϑ between the latter equation and $t^2-(t_1+t_2)t+t_1t_2 = 0$, we shall obtain a sextic in t whose roots are the parameters of the three nodes which the quartic has in general.

If
$$a^2+af+bc-gh = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

the equations $\phi(\vartheta_1, \vartheta_2) = \phi(\vartheta_2, \vartheta_3) = \phi(\vartheta_3, \vartheta_1) = 0$ coexist for an indefinite number of values of ϑ_1, ϑ_2, ϑ_3, and it is evident then that the three nodes of the quartic are replaced by a triple point. Hence we infer that, if a quartic V with a triple point touch a cuspidal cubic U four times, and the stationary tangent of the cubic twice, then an infinite number of triangles can be circumscribed about U and inscribed in V. The quartic U is not limited in any further manner, for such a curve is determined by ten conditions, and we have four constants at our disposal.

If
$$\phi(\vartheta_1) = \phi(\vartheta_2) = \phi(\vartheta_3) = \phi(\vartheta_4),$$

where
$$\phi(\vartheta) = \frac{1}{\vartheta^4}(\vartheta^4-p\vartheta^3+q\vartheta^2-r\vartheta+s),$$

the four corresponding tangents will evidently form a quadrilateral whose six intersections of sides will lie on the same locus. Dividing $\phi(\vartheta_1)-\phi(\vartheta_2) = 0$ by $\vartheta_1-\vartheta_2$, this relation becomes

$$p\vartheta_1^3\vartheta_2^3 - q\vartheta_1^2\vartheta_2^2(\vartheta_1+\vartheta_2)+r\vartheta_1\vartheta_2(\vartheta_1+\vartheta_2)^2-s(\vartheta_1+\vartheta_2)^3-r\vartheta_1^2\vartheta_2^2 + 2s\vartheta_1\vartheta_2(\vartheta_1+\vartheta_2) = 0,$$

or $$u = \frac{-t(rt+2s)}{pt^3+qt^2+rt+s};$$

from which we see that the locus of the intersection of the tangents is the unicursal quartic determined by the equations

$$2x = pt^3+qt^2+rt+s,$$
$$3y = t(pt^3+qt^2-s),$$
$$z = -t^3(rt+2s).$$

Since, from these equations, we have

$$3sy+qz+t(2sx+3ry+pz) = 0,$$

it follows that this quartic must have a triple point determined by $3sy+qz = 0$, $2sx+3ry+pz = 0$. This quartic has an inflexion and corresponding tangent in common with the given cubic, and also touches it in three points elsewhere.

If we consider the case when two sides of the quadrilateral coincide, it is evident that the two other sides will become tangents to the quartic. Thus we see that the six common tangents of the cubic and quartic form three pairs of lines, whose intersection lies on the quartic, and whose chords of contact touch the cubic, the points of contact of the latter lines lying on the quartic.

Let $$\phi(\vartheta) = \frac{1}{\vartheta^3}(\vartheta^3-p\vartheta^2+q\vartheta-r),$$

then the equations $$\phi(\vartheta_1)+\phi(\vartheta_2) = 0,$$
$$\phi(\vartheta_2)+\phi(\vartheta_3) = 0, \quad \phi(\vartheta_3)+\phi(\vartheta_4) = 0, \quad \phi(\vartheta_4)+\phi(\vartheta_1) = 0,$$

being only equivalent to three conditions, will give rise to a series of quadrilaterals, of which the intersections (12), (23), (34), (41) will lie on the same locus. Writing $\phi(\vartheta_1)+\phi(\vartheta_2) = 0$ in the form

$$2\vartheta_1^3\vartheta_2^3 - p\vartheta_1^2\vartheta_2^2(\vartheta_1+\vartheta_2)+q\vartheta_1\vartheta_2(\vartheta_1+\vartheta_2)^2-r(\vartheta_1+\vartheta_2)^3$$
$$-2q\vartheta_1^2\vartheta_2^2+3r\vartheta_1\vartheta_2(\vartheta_1+\vartheta_2) = 0,$$

we get $$u = \frac{-t(2qt+3r)}{2t^3+pt^2+qt+r},$$

from which we see that the locus is a unicursal quartic determined by the equations

$$2x = 2t^3+pt^2+qt+r,$$
$$3y = t(2t^3+pt^2-qt-2r),$$
$$z = -t^3(2qt+3r).$$

This quartic has an inflexion and corresponding tangent in common with the given cubic, and also touches it in three points elsewhere. To find the parameters of the nodes, we have

$$\phi(\vartheta_1)+\phi(\vartheta) = 0, \quad \phi(\vartheta_2)+\phi(\vartheta) = 0, \quad \vartheta_1\vartheta_2+\vartheta(\vartheta_1+\vartheta_2) = 0;$$

hence ϑ_1, ϑ_2 are roots of the equation

$$\{1+\phi(\vartheta)\}\, t^3-pt^2+qt-r=0;$$

and, since $\frac{1}{\vartheta_1}+\frac{1}{\vartheta_2}+\frac{1}{\vartheta}=0$, if t is the third root of this equation, $\frac{1}{\vartheta_1}+\frac{1}{\vartheta_2}+\frac{1}{t}=\frac{q}{r}=\frac{1}{t}-\frac{1}{\vartheta}$, we have $t=\frac{r\vartheta}{r+q\vartheta}$, and ϑ is determined by the cubic

$$(pq-2r)\,\vartheta^3+(q^2+2pr)\,\vartheta^2+qr\vartheta+2r^2=0.$$

If t_1, t_2, t_3 are the roots of this equation, the parameters of the node corresponding to t_1 will be the roots of $t^2-u_1t-u_1t_1=0$; this quartic will, therefore, be in general trinodal.

Since $\phi(\vartheta_1)-\phi(\vartheta_3)=0$, $\phi(\vartheta_2)-\phi(\vartheta_4)=0$,

or $p\vartheta_1^2\vartheta_3^2-q\vartheta_1\vartheta_3(\vartheta_1+\vartheta_3)+r(\vartheta_1^2+\vartheta_3^2+\vartheta_1\vartheta_3)=0$,

the locus of the intersections of the opposite sides of the quadrilateral (13), (24) will be a cuspidal cubic. If we consider the tangents corresponding to the roots of the equation $\vartheta^3\phi(\vartheta)=0$, we see that the points of contact of these tangents and the vertices of the triangle formed by them lie on the quartic.

Of course, corresponding to all the theorems concerning polygons circumscribed about a cuspidal cubic, we shall have, by reciprocation, similar theorems concerning polygons inscribed in a cuspidal cubic.

The Algebraic Solution of the Modular Equation for the Septic Transformation. *By* G. S. Ely, Fellow in Johns Hopkins University, U.S.

[*Read April 6th*, 1882.]

To transform $\frac{dx}{\sqrt{1-x^2\,.\,1-\kappa^2x^2}}=\frac{M\,dy}{\sqrt{1-y^2\,.\,1-\lambda^2y^2}}$,

Prof. Cayley writes for the septic transformation

$$\frac{1-y}{1+y}=\frac{1-x}{1+x}\left(\frac{1-\alpha x+\beta x^2-\gamma x^3}{1+\alpha x+\beta x^2+\gamma x^3}\right)^2,$$

and shows that the conditions necessary for the change of x into

$\frac{1}{\kappa x}$, and y into $\frac{1}{\lambda y}$, are

$$\left.\begin{aligned} \gamma &= \Omega, \\ \kappa^2(\beta^2+2\beta\gamma+2\alpha\gamma) &= \Omega(2\beta+2\alpha+\alpha^2), \\ \kappa^4(2\beta+2\alpha\beta+2\gamma+\alpha^2) &= \Omega(\beta^2+2\alpha\beta+2\gamma+2\alpha\gamma), \\ \kappa^6(1+2\alpha) &= \Omega(\gamma^2+2\beta\gamma), \end{aligned}\right\} \quad \text{.....(I.),}$$

where $\Omega^2 = \frac{\kappa^7}{\lambda}$. Writing $\kappa = u^4, \lambda = v^4$, we have $\Omega = \frac{u^{14}}{v^2}$, $\gamma = \sqrt{\Omega} = \frac{u^7}{v}$, and, by making x and y indefinitely small,

$$M^{-1} = 2\alpha+1 \quad \text{.....(a).}$$

From the fourth of equations (I.),

$$2\beta = u^8v^8M^{-1}-\gamma \quad \text{.....(b),}$$

and from the second and third of equations (I.),

$$v^2(\beta^2+2\beta\gamma+2\alpha\gamma) = u^6(2\beta+2\alpha+\alpha^2) \quad \text{.....(c),}$$

$$u^2v^2(2\beta+2\alpha\beta+2\gamma+\alpha^2) = \beta^2+2\alpha\beta+2\gamma+2\alpha\gamma \quad \text{.....(d).}$$

Then, from equations (a), (b), (c), (d), and the equation $\gamma = \frac{u^7}{v}$, α, β, γ, and M are to be eliminated, and the resultant when reduced will be the modular equation desired.

Eliminating α, β, and γ, we obtain the two equations in M,

$$Av^2M^{-2}-2Bv^2M^{-1}+Cv = 0 \quad \text{.....(e),}$$

$$Bv^4M^{-2}+DvM^{-1}+E = 0 \quad \text{.....(f),}$$

where

$$\begin{aligned} A &= v^8-1, \\ B &= (1+uv)(1-uv)^3, \\ C &= 4u^7-4uv^2+3v(1-u^8), \\ D &= 2u(v^4-u^4)(1+u^2v^2)-2v^3(1-u^8), \\ E &= v^4-u^{12}-6u^5v(1-u^2v^2). \end{aligned}$$

Then, eliminating M^{-2} and M^{-1} from equations (e) and (f), we have the following determinant—

$$\begin{vmatrix} Av^2 & -2Bv^2 & Cv & 0 \\ 0 & Av^2 & -2Bv^2 & Cv \\ Bv^4 & Dv & E & 0 \\ 0 & Bv^4 & Dv & E \end{vmatrix} = 0 = \begin{vmatrix} A & -2Bv & Cv & 0 \\ 0 & A & -2Bv & C \\ Bv^2 & D & E & 0 \\ 0 & Bv^3 & Dv & E \end{vmatrix}$$

$$= A\{AE^2+D(2BEv+CDv)-2BCEv^3\} + B^2v^4(4BE+2DC+C^2v^2) = 0.$$

Substituting the values of A, B, C, D, and E, and expanding, we have

$$\begin{aligned}
&u^{24}+v^{24}-8\,(u^{23}v^7+u^7v^{23}+u^{17}v+uv^{17})+82\,(u^{20}v^4+u^4v^{20})\\
&+20\,(u^{22}v^6+u^6v^{22}+u^{18}v^2+u^2v^{18})-56\,(u^{21}v^5+u^5v^{21}+u^{19}v^3+u^3v^{19})\\
&+64\,(u^{21}v^{13}+u^{13}v^{21}+u^{11}v^3+u^3v^{11})-208\,(u^{20}v^{12}+u^{12}v^{20}+u^{12}v^4+u^4v^{12})\\
&+352\,(u^{19}v^{11}+u^{11}v^{19}+u^{13}v^5+u^5v^{13})-272\,(u^{18}v^{10}+u^{10}v^{18}+u^{14}v^6+u^6v^{14})\\
&-160\,(u^{17}v^9+u^9v^{17}+u^{15}v^7+u^7v^{15})+463\,(u^{16}v^8+u^8v^{16})\\
&-128\,(u^5v^5+u^{19}v^{19})+448\,(u^6v^6+u^{18}v^{18})-512\,(u^7v^7+u^{17}v^{17})\\
&-224\,(u^8v^8+u^{16}v^{16})+1232\,(u^9v^9+u^{15}v^{15})-968\,(u^{10}v^{10}+u^{14}v^{14})\\
&-976\,(u^{11}v^{11}+u^{13}v^{13})+2236u^{12}v^{12}=0 \quad \text{.....(II.)}
\end{aligned}$$

In the *Philosophical Transactions* for 1874, p. 397, Prof. Cayley has for the septic transformation these equations

$$\left.\begin{aligned}
\kappa^3\alpha^2 &= \Omega\delta^2,\\
\kappa\,(2\alpha\gamma+2\alpha\beta+\beta^2) &= \Omega\,(\gamma^2+2\gamma\delta+2\beta\delta),\\
\gamma^2+2\beta\gamma+2\alpha\delta+2\beta\delta &= \Omega\kappa\,(2\alpha\gamma+2\beta\gamma+2\alpha\delta+\beta^2),\\
\delta^2+2\gamma\delta &= \Omega\kappa^3\,(\alpha^2+2\alpha\beta),
\end{aligned}\right\} \quad \text{.....(III.)}$$

Apparently there is one more constant in these equations than in equations (I.) But if in these equations we divide by $\alpha^2\Omega$, and then make $\frac{\beta}{\alpha}=a$, $\frac{\gamma}{\alpha}=\beta$, $\frac{\delta}{\alpha}=\gamma$, and $\frac{\kappa^3}{\Omega}=\Omega$, we obtain equations (I.) Now, in equations (III.), making $\alpha=0$ and $\delta=0$, the equations are

$$\kappa\beta^2=\Omega\gamma^2,$$

$$\gamma^2+2\beta\gamma=\Omega\kappa\,(2\beta\gamma+\beta^2);$$

whence dividing by $\beta^2\Omega$, and making $\frac{\kappa}{\Omega}=\Omega$, $\frac{\gamma}{\beta}=a$, we obtain

$$a^2=\Omega,$$

$$\kappa^2\,(2a+1)=\Omega\,(a^2+2a),$$

which are exactly the equations for the cubic transformation. Therefore equation (II.) is divisible by the left-hand member of the modular equation for the cubic transformation, that is, by

$$u^4-v^4+2uv\,(1-u^2v^2).$$

Again, in equations (III.), making $\gamma=\kappa a$, $\delta=\kappa\beta$, the equations become

$$\begin{aligned}
\kappa a^2 &= \Omega\beta^2,\\
2\kappa^2a^2+\kappa\,(2a\beta+\beta^2) &= \Omega\kappa^2\,(a^2+2a\beta)+2\Omega\kappa\beta^2,\\
\kappa^2a^2+2\kappa\,(\beta^2+2a\beta) &= 2\Omega\kappa^2\,(a^2+2a\beta)+\Omega\kappa\beta^2,\\
\beta^2+2a\beta &= \Omega\kappa\,(a^2+2a\beta).
\end{aligned}$$

The first and last of these being satisfied, the second and third are satisfied identically. Divide the first and last by $a^2\Omega$, and make $\frac{\kappa}{\Omega} = \Omega$ and $\frac{\beta}{a} = a$; whence

$$a^2 = \Omega,$$

$$\kappa^2 (2a+1) = \Omega (a^2+2a).$$

Therefore equation (II.) is again divisible by

$$u^4-v^4+2uv (1-u^2v^2).$$

But we observe that equation (II.) is symmetrical in u and v, and therefore, as it is divisible by

$$\{u^4-v^4+2uv (1-u^2v^2)\}^2,$$

it must also be divisible by

$$\{v^4-u^4+2uv (1-u^2v^2)\}^2.$$

Then, actually performing the division, we find equation (II.) to be

$$\{u^4-v^4+2uv (1-u^2v^2)\}^2 \{v^4-u^4+2uv (1-u^2v^2)\}^2$$
$$\times \{(1-u^8) (1-v^8)-(1-uv)^8\} = 0.$$

This is the product of five factors, and, as the first four belong to cubic transformations, the last only can belong to the septic transformation. We have thus, as the modular equation for the septic transformation,

$$(1-u^8) (1-v^8) = (1-uv)^8,$$

a result which I believe has never been obtained before algebraically.

On a Symmetric Determinant connected with Lagrange's Interpolation Problem. *By* Thomas Muir, M.A., F.R.S.E.

[*Read April 6th*, 1882.]

1. In Lagrange's interpolation-problem it is required to find the values of $A_1, A_2, \ldots A_n$, so that

$$A_1 \sin x + A_2 \sin 2x + \ldots + A_n \sin nx$$

may coincide in value with an assigned function when x has the values $\pi/(n+1), 2\pi/(n+1), \ldots, n\pi/(n+1)$. The data lead to a set of n equa-

tions, the determinant of which is

$$\begin{vmatrix} \sin\frac{\pi}{n+1}, & \sin\frac{2\pi}{n+1}, & \sin\frac{3\pi}{n+1}, & \ldots\ldots, & \sin\frac{n\pi}{n+1} \\ \sin\frac{2\pi}{n+1}, & \sin\frac{4\pi}{n+1}, & \sin\frac{6\pi}{n+1}, & \ldots\ldots, & \sin\frac{2n\pi}{n+1} \\ \sin\frac{3\pi}{n+1}, & \sin\frac{6\pi}{n+1}, & \sin\frac{9\pi}{n+1}, & \ldots\ldots, & \sin\frac{3n\pi}{n+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \sin\frac{n\pi}{n+1}, & \sin\frac{2n\pi}{n+1}, & \sin\frac{3n\pi}{n+1}, & \ldots\ldots, & \sin\frac{n^2\pi}{n+1} \end{vmatrix}$$

or $$\left| \sin\frac{1\,.\,1\pi}{n+1}, \sin\frac{2\,.\,2\pi}{n+1}, \sin\frac{3\,.\,3\pi}{n+1}, \ldots\ldots, \sin\frac{n\,.\,n\pi}{n+1} \right| .$$

Certain of the properties of this symmetric determinant, L_n say, seem well worthy of attention.

2. It is readily made apparent that the number of different elements in the determinant—different, that is to say, in magnitude—is $\frac{1}{2}n$ or $\frac{1}{2}(n+3)$, according as n is even or odd; also, that with respect to the secondary diagonal the determinant is alternately symmetric and skew; and that with respect to the bisecting line parallel to the columns it has the same property.

3. Multiplying each element of the s^{th} column by $\sin\frac{rs\pi}{n+1}$, and increasing the product by $\sin\frac{r(s-1)\pi}{n+1}$ times the corresponding element of the $(s-1)^{\text{th}}$ column, by $\sin\frac{r(s-2)\pi}{n+1}$ times the corresponding element of the $(s-2)^{\text{th}}$ column, and so on with the corresponding elements of all the other columns, we obtain a new determinant equal to

$$L_n \times \sin\frac{rs\pi}{n+1}.$$

This new determinant, however, has all the elements of its s^{th} column equal to zero, except the element in the r^{th} row, which equals $\frac{1}{2}(n+1)$, in virtue of the trigonometrical theorem that

$$\sin\frac{r\pi}{n+1}\sin\frac{s\pi}{n+1} + \sin\frac{2r\pi}{n+1}\sin\frac{2s\pi}{n+1} + \ldots + \sin\frac{nr\pi}{n+1}\sin\frac{ns\pi}{n+1}$$

has the value 0 or $\frac{1}{2}(n+1)$ according as r is unequal or equal to s. It thus results that

$$L_n \times \sin\frac{rs\pi}{n+1} = \tfrac{1}{2}(n+1) \times \text{cofactor of } \sin\frac{rs\pi}{n+1},$$

and therefore that $$\frac{\sin\frac{rs\pi}{n+1}}{\text{cofactor of } \sin\frac{rs\pi}{n+1}} = \frac{\frac{1}{2}(n+1)}{L_n} \quad \text{.....(A),}$$

that is to say, is equal to a constant with respect to r and s. Consequently we have the theorem—

Any two elements of

$$\left| \sin\frac{1.1\pi}{n+1},\ \sin\frac{2.2\pi}{n+1},\ \ldots\ldots,\ \sin\frac{n.n\pi}{n+1} \right|$$

are proportional to their cofactors in the determinant .....(A').

4. Multiplying L_n by itself, and using the trigonometrical theorem before mentioned, we obtain a determinant having each element in the principal diagonal equal to $\frac{1}{2}(n+1)$, and every other element equal to 0. Hence $$(L_n)^2 = \{\tfrac{1}{2}(n+1)\}^n,$$

and therefore

$$\left| \sin\frac{1.1\pi}{n+1},\ \sin\frac{2.2\pi}{n+1},\ \ldots\ldots,\ \sin\frac{n.n\pi}{n+1} \right| = \{\tfrac{1}{2}(n+1)\}^{\frac{1}{2}n} \text{.....(B).}$$

The sign-factor to be taken on the right, when the square root is extracted, will be seen from what follows to be $(-1)^{\frac{1}{2}n(n-1)}$.

From this and (A) we have the evident deduction that the complementary minor of any element, $\sin\frac{rs\pi}{n+1}$, in L_n is equal to

$$(-1)^{\frac{1}{2}n(n-1)+r+s} \sin\frac{rs\pi}{n+1} \cdot \{\tfrac{1}{2}(n+1)\}^{\frac{1}{2}n-1}.$$

5. Taking now a determinant of the $2m^{\text{th}}$ order, and increasing each element of the first row by the corresponding element of the last row, each element of the second row by the corresponding element of the second row from the end, and so on, we have

$$L_{2m} = \begin{vmatrix} 2\sin\frac{\pi}{2m+1}, & 0, & 2\sin\frac{3\pi}{2m+1}, & \ldots\ldots, & 0 \\ 2\sin\frac{2\pi}{2m+1}, & 0, & 2\sin\frac{6\pi}{2m+1}, & \ldots\ldots, & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 2\sin\frac{m\pi}{2m+1}, & 0, & 2\sin\frac{3m\pi}{2m+1}, & \ldots\ldots, & 0 \\ \sin\frac{(m+1)\pi}{2m+1}, & \sin\frac{(m+1)2\pi}{2m+1}, & \sin\frac{(m+1)3\pi}{2m+1}, & \ldots\ldots, & \sin\frac{(m+1)2m\pi}{2m+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \sin\frac{2m\pi}{2m+1}, & \sin\frac{2m.2\pi}{2m+1}, & \sin\frac{2m.3\pi}{2m+1}, & \ldots\ldots, & \sin\frac{2m.2m\pi}{2m+1} \end{vmatrix},$$

for $\sin\frac{rs\pi}{2m+1}+\sin\frac{(2m-r+1)\,s\pi}{2m+1}$

$$=\sin\frac{rs\pi}{2m+1}+(-1)^{s-1}\sin\left(s\pi-\frac{2m-r+1}{2m+1}s\pi\right)$$

$$=\sin\frac{rs\pi}{2m+1}+(-1)^{s-1}\sin\frac{rs\pi}{2m+1}$$

$$=2\sin\frac{rs\pi}{2m+1}\text{ or }0$$

according as s is odd or even. This determinant, however, is expressible as the product of two complementary minors, viz.,

$$L_{2m}=(-1)^{\frac{1}{2}m(3m+1)}\,2^m\begin{vmatrix}\sin\frac{\pi}{2m+1}, & \sin\frac{3\pi}{2m+1}, & \ldots\ldots, & \sin\frac{(2m-1)\,\pi}{2m+1}\\ \sin\frac{2\pi}{2m+1}, & \sin\frac{6\pi}{2m+1}, & \ldots\ldots, & \sin\frac{(2m-1)\,2\pi}{2m+1}\\ \ldots & \ldots & \ldots & \ldots\\ \sin\frac{m\pi}{2m+1}, & \sin\frac{3m\pi}{2m+1}, & \ldots\ldots, & \sin\frac{(2m-1)\,m\pi}{2m+1}\end{vmatrix}$$

$$\times\begin{vmatrix}\sin\frac{(m+1)\,2\pi}{2m+1}, & \sin\frac{(m+1)\,4\pi}{2m+1}, & \ldots\ldots, & \sin\frac{(m+1)\,2m\pi}{2m+1}\\ \sin\frac{(m+2)\,2\pi}{2m+1}, & \sin\frac{(m+2)\,4\pi}{2m+1}, & \ldots\ldots, & \sin\frac{(m+2)\,2m\pi}{2m+1}\\ \ldots & \ldots & \ldots & \ldots\\ \sin\frac{2m\,.\,2\pi}{2m+1}, & \sin\frac{2m\,.\,4\pi}{2m+1}, & \ldots\ldots, & \sin\frac{2m\,.\,2m\pi}{2m+1}\end{vmatrix}.$$

But the element in the s^{th} row and r^{th} column of the second determinant

here $$=\sin\frac{(m+s)\,2r\pi}{2m+1}$$

$$=(-1)^{r-1}\sin\left(r\pi-\frac{m+s}{2m+1}2r\pi\right)$$

$$=(-1)^{r-1}\sin\frac{(2s-1)\,r\pi}{2m+1},$$

that is to say, is equal to the element in the s^{th} column and r^{th} row of the first determinant when r is even, and differs from the said element only in sign when r is odd. Altering, therefore, the signs of all the elements in the odd-numbered columns of the second determinant, that

is, multiplying it by $(-1)^{\frac{1}{2}m(m+1)}$, we have

$$L_{2m} = (-1)^m 2^m \begin{vmatrix} \sin\frac{\pi}{2m+1}, & \sin\frac{3\pi}{2m+1}, & \ldots\ldots, & \sin\frac{(2m-1)\pi}{2m+1} \\ \sin\frac{2\pi}{2m+1}, & \sin\frac{6\pi}{2m+1}, & \ldots\ldots, & \sin\frac{(2m-1)2\pi}{2m+1} \\ \ldots & \ldots & \ldots & \ldots \\ \sin\frac{m\pi}{2m+1}, & \sin\frac{3m\pi}{2m+1}, & \ldots\ldots, & \sin\frac{(2m-1)m\pi}{2m+1} \end{vmatrix}^2$$

or

$$\begin{vmatrix} \sin\frac{1\,.\,1\pi}{2m+1}, & \sin\frac{2\,.\,2\pi}{2m+1}, & \ldots\ldots, & \sin\frac{2m\,.\,2m\pi}{2m+1} \end{vmatrix}$$

$$= (-1)^m 2^m \begin{vmatrix} \sin\frac{1\,.\,1\pi}{2m+1}, & \sin\frac{2\,.\,3\pi}{2m+1}, & \ldots\ldots, & \sin\frac{m\,(2m-1)\,\pi}{2m+1} \end{vmatrix}^2 \quad \ldots(C).$$

From this result and (B) we obtain another, viz.,

$$\begin{vmatrix} \sin\frac{1\,.\,1\pi}{2m+1}, & \sin\frac{2\,.\,3\pi}{2m+1}, & \ldots\ldots, & \sin\frac{m\,(2m-1)\,\pi}{2m+1} \end{vmatrix} = \{\tfrac{1}{4}(2m+1)\}^{\frac{1}{2}m} \quad \ldots\ldots\ldots\ldots\ldots\ldots(D).$$

6. If now we take the determinant in the right-hand member of (C), and reverse the order of the columns,—a process equivalent to multiplying the determinant by $(-1)^{\frac{1}{2}m(m-1)}$,—the element in the r^{th} row and s^{th} column will be

$$\sin\frac{r\,(2m-2s+1)\,\pi}{2m+1}.$$

But this is equal to

$$(-1)^{r-1}\sin\left\{r\pi - \frac{r\,(2m-2s+1)\,\pi}{2m+1}\right\}$$

$$= (-1)^{r-1}\sin\frac{rs\,.\,2\pi}{2m+1}.$$

Substituting the latter form in place of the original, and altering the signs of all the elements in the even-numbered rows, that is, multiplying again by $(-1)^{\frac{1}{2}m(m-1)}$, we obtain the curious identity

$$\begin{vmatrix} \sin\frac{1\,.\,1\pi}{2m+1}, & \sin\frac{2\,.\,3\pi}{2m+1}, & \ldots\ldots, & \sin\frac{m\,(2m-1)\,\pi}{2m+1} \end{vmatrix} = \begin{vmatrix} \sin\frac{1\,.\,1\,.\,2\pi}{2m+1}, & \sin\frac{2\,.\,2\,.\,2\pi}{2m+1}, & \ldots\ldots, & \sin\frac{m\,.\,m\,.\,2\pi}{2m+1} \end{vmatrix} \quad \ldots\ldots(E).$$

The value of the new determinant on the right is given by (D), and (C) assumes a more elegant form by the substitution which is now possible.

7. The determinant L_n is resolvable into two determinant factors in the case also when n is odd. Little interest, however, attaches to this case in comparison with the other.

8. The investigation of the determinant which differs from L_n in having *cosines* everywhere instead of *sines* leads to results more or less closely analogous with the foregoing. Thus we have

$$\left.\begin{array}{l} \left| \cos\frac{1.1\pi}{n+1},\ \cos\frac{2.2\pi}{n+1},\ \ldots\ldots,\ \cos\frac{n.n\pi}{n+1} \right| \\ \quad = (-1)^{\frac{1}{2}n}\,\tfrac{1}{2}\{\tfrac{1}{2}(n+1)\}^{\frac{1}{2}n-1} \qquad (n \text{ even}) \\ \quad = 0 \qquad\qquad\qquad\qquad\quad (n \text{ odd}) \end{array}\right\} \ldots\ldots(\mathrm{F}),$$

$$\left| \cos\frac{1.1\pi}{2m+1},\ \cos\frac{2.2\pi}{2m+1},\ \ldots\ldots,\ \cos\frac{2m.2m\pi}{2m+1} \right|$$

$$= (-1)^m 2^m \left| \cos\frac{1.1\pi}{2m+1},\ \cos\frac{2.3\pi}{2m+1},\ \ldots\ldots,\ \cos\frac{m(2m-1)\pi}{2m+1} \right|^2 \ldots(\mathrm{G}),$$

$$\left| \cos\frac{1.1\pi}{2m+1},\ \cos\frac{2.3\pi}{2m+1},\ \ldots\ldots,\ \cos\frac{m(2m-1)\pi}{2m+1} \right| = \tfrac{1}{2}\{\tfrac{1}{4}(2m+1)\}^{\frac{m-1}{2}} \ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{H}),$$

and

$$\left| \cos\frac{1.1\pi}{2m+1},\ \cos\frac{2.3\pi}{2m+1},\ \ldots\ldots,\ \cos\frac{m(2m-1)\pi}{2m+1} \right|$$

$$= (-1)^m \left| \cos\frac{1.1.2\pi}{2m+1},\ \cos\frac{2.2.2\pi}{2m+1},\ \ldots\ldots,\ \cos\frac{m.m.2\pi}{2m+1} \right| \ldots(\mathrm{I}).$$

Note on the Condensation of Skew Determinants, which are partially zero-axial. By Thomas Muir, M.A., F.R.S.E.

[*Read April 6th*, 1882.]

1. Taking, in the first place, a skew determinant having a zero only in the last place of the principal diagonal, *e.g.*,

$$\begin{vmatrix} x_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ -a_2 & x_2 & b_3 & b_4 & b_5 & b_6 \\ -a_3 & -b_3 & x_3 & c_4 & c_5 & c_6 \\ -a_4 & -b_4 & -c_4 & x_4 & d_5 & d_6 \\ -a_5 & -b_5 & -c_5 & -d_5 & x_5 & e_6 \\ -a_6 & -b_6 & -c_6 & -d_6 & -e_6 & 0 \end{vmatrix}, \text{ or } D_1,$$

we see that it is clearly equal to

$$\frac{1}{a_6^4}\begin{vmatrix} x_1 & a_2a_6 & a_3a_6 & a_4a_6 & a_5a_6 & a_6 \\ -a_2 & x_2a_6 & b_3a_6 & b_4a_6 & b_5a_6 & b_6 \\ -a_3 & -b_3a_6 & x_3a_6 & c_4a_6 & c_5a_6 & c_6 \\ -a_4 & -b_4a_6 & -c_4a_6 & x_4a_6 & d_5a_6 & d_6 \\ -a_5 & -b_5a_6 & -c_5a_6 & -d_5a_6 & x_5a_6 & e_6 \\ -a_6 & -b_6a_6 & -c_6a_6 & -d_6a_6 & -e_6a_6 & 0 \end{vmatrix}.$$

Multiplying the latter by x_1 in the form

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_1 \\ -b_6 & 1 & 0 & 0 & 0 & -a_2 \\ -c_6 & 0 & 1 & 0 & 0 & -a_3 \\ -d_6 & 0 & 0 & 1 & 0 & -a_4 \\ -e_6 & 0 & 0 & 0 & 1 & -a_5 \end{vmatrix},$$

we obtain

$$D_1 = \frac{1}{x_1 a_6^4}\begin{vmatrix} x_1 & a_6x_1 & -b_6x_1 & -c_6x_1 & -d_6x_1 & -e_6x_1 \\ -a_2 & b_6x_1 & a_6x_2 & \|\, a_2b_3c_6 \,\| & \|\, a_2b_4d_6 \,\| & \|\, a_2b_5c_6 \,\| \\ -a_3 & c_6x_1 & -\|\, a_2b_3c_6 \,\| & a_6x_3 & \|\, a_3c_4d_6 \,\| & \|\, a_3c_5e_6 \,\| \\ -a_4 & d_6x_1 & -\|\, a_2b_4d_6 \,\| & -\|\, a_3c_4d_6 \,\| & a_6x_4 & \|\, a_4d_5e_6 \,\| \\ -a_5 & e_6x_1 & -\|\, a_2b_5e_6 \,\| & -\|\, a_3c_5e_6 \,\| & -\|\, a_4d_5e_6 \,\| & a_6x_5 \\ -a_6 & 0 & 0 & 0 & 0 & 0 \end{vmatrix},$$

where the Pfaffian notation employed is explained by the example

$$\|\, a_2\ b_3\ c_6 \,\| \equiv \begin{vmatrix} a_2 & a_3 & a_6 \\ & b_3 & b_6 \\ & & c_6 \end{vmatrix} = a_2c_6 - a_3b_6 + a_6b_3.$$

We thus have the result

$$x_1a_6^3 D_1 = \begin{vmatrix} a_6x_1 & b_6x_1 & c_6x_1 & d_6x_1 & e_6x_1 \\ -b_6x_1 & a_6x_2 & \|\, a_2b_3c_6 \,\| & \|\, a_2b_4d_6 \,\| & \|\, a_2b_5e_6 \,\| \\ -c_6x_1 & -\|\, a_2b_3c_6 \,\| & a_6x_3 & \|\, a_3c_4d_6 \,\| & \|\, a_3c_5e_6 \,\| \\ -d_6x_1 & -\|\, a_2b_4d_6 \,\| & -\|\, a_3c_4d_6 \,\| & a_6x_4 & \|\, a_4d_5e_6 \,\| \\ -e_6x_1 & -\|\, a_2b_5e_6 \,\| & -\|\, a_3c_5e_6 \,\| & -\|\, a_4d_5e_6 \,\| & a_6x_5 \end{vmatrix} \quad \ldots(\text{I.}),$$

where it has to be noted that the determinant on the right is still skew with respect to the principal diagonal.

2. Secondly, let us take a skew determinant having the last *two* elements of the principal diagonal each equal to zero, *e.g.*, the deter-

minant, D_2 say, got from D_1 by making $x_5 = 0$. From (I) we have at the outset

$$x_1 a_6^3 D_2 = \begin{vmatrix} a_6x_1 & b_6x_1 & c_6x_1 & d_6x_1 & e_6x_1 \\ -b_6x_1 & a_6x_2 & \| a_2b_3c_6 | & \| a_2b_4d_6 | & \| a_2b_5e_6 | \\ -c_6x_1 & -\| a_2b_3c_6 | & a_6x_3 & \| a_3c_4d_6 | & \| a_3c_5e_6 | \\ -d_6x_1 & -\| a_2b_4d_6 | & -\| a_3c_4d_6 | & a_6x_4 & \| a_4d_5e_6 | \\ -e_6x_1 & -\| a_2b_5e_6 | & -\| a_3c_5e_6 | & -\| a_4d_5e_6 | & 0 \end{vmatrix}$$

But the determinant on the right is exactly of the form dealt with in § 1, and thus we know that it is transformable into

$$\frac{1}{a_6x_1(e_6x_1)^2} \begin{vmatrix} a_6e_6x_1^2 & a_6x_1 \| a_2b_5e_6 | & a_6x_1 \| a_3c_5e_6 | & a_6x_1 \| a_4d_5e_6 | \\ -a_6x_1 \| a_2b_5e_6 | & a_6e_6x_1x_2 & P & Q \\ -a_6x_1 \| a_3c_5e_6 | & -P & a_6e_6x_1x_3 & R \\ -a_6x_1 \| a_4d_5e_6 | & -Q & -R & a_6e_6x_1x_4 \end{vmatrix} \quad \ldots\ldots\ldots\ldots(\omega),$$

where

$$P = \begin{vmatrix} b_6x_1 & c_6x_1 & e_6x_1 \\ & \| a_2b_3e_6 | & \| a_2b_5e_6 | \\ & & \| a_3c_5e_6 | \end{vmatrix}, \quad Q = \begin{vmatrix} b_6x_1 & d_6x_1 & e_6x_1 \\ & \| a_2b_4d_6 | & \| a_2b_5e_6 | \\ & & \| a_4d_5e_6 | \end{vmatrix},$$

and

$$R = \begin{vmatrix} c_6x_1 & d_6x_1 & e_6x_1 \\ & \| a_3c_4d_6 | & \| a_3c_5e_6 | \\ & & \| a_4d_5e_6 | \end{vmatrix}.$$

Now it is readily shown that these compound Pfaffians are equal to

$$x_1a_6 \| b_3c_5e_6 |, \quad x_1a_6 \| b_4d_5e_6 |, \quad x_1a_6 \| c_4d_5e_6 |,$$

respectively; hence (ω) becomes

$$\frac{a_6^3}{e_6^2 x_1} \begin{vmatrix} e_6 & \| a_2b_5e_6 | & \| a_3c_5e_6 | & \| a_4d_5e_6 | \\ -\| a_2b_5e_6 | & e_6x_1x_2 & x_1 \| b_3c_5e_6 | & x_1 \| b_4d_5e_6 | \\ -\| a_3c_5e_6 | & -x_1 \| b_3c_5e_6 | & e_6x_1x_3 & x_1 \| c_4d_5e_6 | \\ -\| a_4d_5e_6 | & -x_1 \| b_4d_5e_6 | & -x_1 \| c_4d_5e_6 | & e_6x_1x_4 \end{vmatrix};$$

and thus we almost immediately obtain the result

$$e_6^2 D_2 = \begin{vmatrix} e_6x_1 & \| a_2b_5e_6 | & \| a_3c_5e_6 | & \| a_4d_5e_6 | \\ -\| a_2b_5e_6 | & e_6x_2 & \| b_3c_5e_6 | & \| b_4d_5e_6 | \\ -\| a_3c_5e_6 | & -\| b_3c_5e_6 | & e_6x_3 & \| c_4d_5e_6 | \\ -\| a_4d_5e_6 | & -\| b_4d_5e_6 | & -\| c_4d_5e_6 | & e_6x_4 \end{vmatrix} \quad \ldots\ldots(\text{II.})$$

3. Putting $x_1 = x_2 = x_3 = x_4 = 0$ in the foregoing, and extracting the

square root, we have

$$e_6 \left| \begin{matrix} a_2 & a_3 & a_4 & a_5 & a_6 \\ & b_3 & b_4 & b_5 & b_6 \\ & & c_4 & c_5 & c_6 \\ & & & d_5 & d_6 \\ & & & & e_6 \end{matrix} \right| = \left| \begin{matrix} \| a_2 b_5 e_6 \| & \| a_3 c_5 e_6 \| & \| a_4 d_5 e_6 \| \\ & \| b_3 c_5 e_6 \| & \| b_4 d_5 e_6 \| \\ & & \| c_4 d_5 e_6 \| \end{matrix} \right|$$

an instance of a theorem in Pfaffians corresponding to the well-known theorem in determinants which gives the relation between a determinant and a minor of the adjugate determinant.

4. Proceeding now to the third case, let us denote by D_3 the determinant obtained from D_1 by making $x_4 = x_5 = 0$. To begin with, we have from (II.), if we write $F_1, F_2, \ldots F_6$ for the Pfaffians there occurring,

$$e_6^2 D_3 = \left| \begin{matrix} e_6 x_1 & F_1 & F_2 & F_3 \\ -F_1 & e_6 x_2 & F_4 & F_5 \\ -F_2 & -F_4 & e_6 x_3 & F_6 \\ -F_3 & -F_5 & -F_6 & 0 \end{matrix} \right|.$$

But this determinant, again, is of the form considered in § 1; therefore, from (I.), we have

$$e_6^2 D_3 = \left| \begin{matrix} e_6 x_1 F_3 & e_6 x_1 F_5 & e_6 x_1 F_6 \\ -e_6 x_1 F_5 & e_6 x_2 F_3 & \Phi \\ -e_6 x_1 F_6 & -\Phi & e_6 x_3 F_3 \end{matrix} \right| \div e_6 x_1 F_3 \ldots\ldots\ldots\ldots (\xi),$$

where

$$\Phi = \left| \begin{matrix} F_1 & F_2 & F_3 \\ & F_4 & F_5 \\ & & F_6 \end{matrix} \right|.$$

Now the compound Pfaffian just written has been shown (§ 3) to be equal to $e_6 \| a_2 b_3 c_4 d_5 e_6 \|$.

Making, therefore, this substitution in (ξ), and simplifying, we obtain

$$x_1 \| a_4 d_5 e_6 \| D_3 = \left| \begin{matrix} x_1 \| a_4 d_5 e_6 \| & x_1 \| b_4 d_5 e_6 \| & x_1 \| c_4 d_5 e_6 \| \\ -x_1 \| b_4 d_5 e_5 \| & x_2 \| a_4 d_5 e_6 \| & \| a_2 b_3 c_4 d_5 e_6 \| \\ -x_1 \| c_4 d_5 e_6 \| & -\| a_2 b_3 c_4 d_5 e_6 \| & x_3 \| a_4 d_5 e_6 \| \end{matrix} \right| \ldots (\text{III.})$$

5. In a similar manner and with a similar notation, it is found that

$$D_4 = \left| \begin{matrix} x_1 \| c_4 d_5 e_6 \| & \| a_2 b_3 c_4 d_5 e_6 \| \\ -\| a_2 b_3 c_4 d_5 e_6 \| & x_2 \| c_4 d_5 e_6 \| \end{matrix} \right| \ldots\ldots\ldots\ldots (\text{IV.}),$$

a result readily verifiable otherwise. When we come to D_5 and D_6, we find ourselves upon ground already explored.

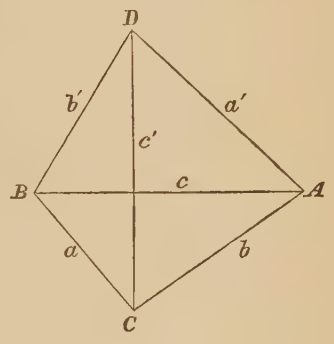
On the General Equation of the Second Degree referred to Tetrahedral Coordinates. *By the* Rev. A. J. C. Allen, M.A., Fellow and Assistant Tutor of Peterhouse, Cambridge.

[*Read April 6th*, 1882.]

Most of the results in the following paper, notably equations (6), (9), and (12), were obtained before my attention was drawn to the paper by Mr. Hart on the same subject, in Vol. xii. of the *Proceedings* of the Society, p. 135; as, however, the methods of the two papers are entirely different, and I have obtained some results beyond those of Mr. Hart, I think the paper may be not without interest to the Society.

I have adopted Mr. Hart's notation throughout, in order to facilitate comparison, and because it is simpler than that which I had myself employed before I saw his paper.

1. Let $ABCD$ be the fundamental tetrahedron, and let us denote the six edges BC, CA, AB, AD, BD, CD by a, b, c, a', b', c'.



Let r be the distance between any two points $\alpha\beta\gamma\delta$, $\alpha'\beta'\gamma'\delta'$; then we may put

$$r^2 = A(\alpha-\alpha')^2 + B(\beta-\beta')^2 + C(\gamma-\gamma')^2 + D(\delta-\delta')^2 + E(\alpha-\alpha')(\beta-\beta') + \ldots,$$

where A, $B\ldots$ are constants. But

$$\alpha-\alpha'+\beta-\beta'+\gamma-\gamma'+\delta-\delta' = 0,$$

therefore

$$(\alpha-\alpha')^2 = -(\alpha-\alpha')(\beta-\beta') - (\alpha-\alpha')(\gamma-\gamma') - (\alpha-\alpha')(\delta-\delta'),$$

and so on; hence we may write

$$r^2 = A(\alpha-\alpha')(\beta-\beta') + B(\alpha-\alpha')(\gamma-\gamma') + C(\alpha-\alpha')(\delta-\delta') + D(\beta-\beta')(\gamma-\gamma') + E(\beta-\beta')(\delta-\delta') + F(\gamma-\gamma')(\delta-\delta'),$$

where A, $B\ldots$ are constants.

To determine these, assume $\alpha=\beta=\gamma=0$, $\beta'=\gamma'=\delta'=0$,

therefore

$$a'^2 = -C;$$

similarly, $b'^2 = -E$, $c'^2 = -F$, $a^2 = -D$, $b^2 = -B$, $c^2 = -A$,

$$\therefore\quad r^2 = a^2(\beta-\beta')(\gamma'-\gamma)+b^2(\alpha-\alpha')(\gamma'-\gamma)+c^2(\alpha-\alpha')(\beta'-\beta)$$
$$+c'^2(\gamma'-\gamma)(\delta-\delta')+b'^2(\beta'-\beta)(\delta-\delta')+a'^2(\alpha-\alpha')(\delta'-\delta)\ldots(1).$$

2. Let $$\frac{\alpha-\alpha_0}{\lambda}=\frac{\beta-\beta_0}{\mu}=\frac{\gamma-\gamma_0}{\nu}=\frac{\delta-\delta_0}{\rho}=r,$$

where r is the distance from $\alpha\beta\gamma\delta$ to $\alpha_0\beta_0\gamma_0\delta_0$, be the equation to any line through the latter point. We have

$$\lambda+\mu+\nu+\rho=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and $$\frac{a^2(\beta-\beta_0)(\gamma-\gamma_0)+\ldots\ldots}{a^2.\mu\nu+b^2.\lambda\nu+c^2.\lambda\mu+a'^2.\lambda\rho+b'^2.\mu\rho+c'^2.\nu\rho}=r^2,$$

therefore

$$a^2.\mu\nu+b^2.\lambda\nu+c^2.\lambda\mu+a'^2.\lambda\rho+b'^2.\mu\rho+c'^2.\nu\rho=-1\ldots\ldots\ldots(3).$$

3. Let $$u_{11}\alpha^2+u_{22}\beta^2+u_{33}\gamma^2+u_{44}\delta^2+2u_{12}\alpha\beta+2u_{13}\alpha\gamma+2u_{14}\alpha\delta$$
$$+2u_{23}\beta\gamma+2u_{24}\beta\delta+2u_{34}\gamma\delta\equiv\phi(\alpha\beta\gamma\delta)=0$$

be the equation to any quadric, and let $\alpha_0\beta_0\gamma_0\delta_0$ be the coordinates of the centre, and r the semi-diameter whose direction is defined by the ratios $\lambda:\mu:\nu:\rho$. We have at once

$$r^2=-\frac{\phi(\alpha_0\beta_0\gamma_0\delta_0)}{\varphi(\lambda\mu\nu\rho)} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

since $\alpha_0\beta_0\gamma_0\delta_0$ satisfy $$\frac{d\phi}{d\alpha_0}=\frac{d\phi}{d\beta_0}=\frac{d\phi}{d\gamma_0}=\frac{d\phi}{d\delta_0},$$

and $$\lambda+\mu+\nu+\rho=0,$$

therefore r^2 is a maximum or a minimum when $\phi(\lambda\mu\nu\rho)$ is a minimum or a maximum, λ, μ, ν, ρ being connected by the equations (2) and (3).

For a maximum or minimum value of $\phi(\lambda\mu\nu\rho)$, we have

$$\left.\begin{array}{c}(u_{11}\lambda+u_{12}\mu+u_{13}\nu+u_{14}\rho)\,\delta\lambda+(u_{12}\lambda+u_{22}\mu+u_{23}\nu+u_{24}\rho)\,\delta\mu\\+(u_{13}\lambda+u_{23}\mu+u_{33}\nu+u_{34}\rho)\,\delta\nu+(u_{14}\lambda+u_{24}\mu+u_{34}\nu+u_{44}\rho)\,\delta\rho=0\\ \delta\lambda+\delta\mu+\delta\nu+\delta\rho=0\\ (b^2\nu+c^2\mu+a'^2\rho)\,\delta\lambda+(a^2\nu+c^2\lambda+b'^2\rho)\,\delta\mu\\+(b^2\lambda+a^2\mu+c'^2\rho)\,\delta\nu+(a'^2\lambda+b'^2\mu+c'^2\nu)\,\delta\rho=0\end{array}\right\}\ldots($$

therefore
$$u_{11}\lambda+u_{12}\mu+u_{13}\nu+u_{14}\rho+k(c^2\mu+b^2\nu+a'^2\rho)+k'=0,$$
$$u_{12}\lambda+u_{22}\mu+u_{23}\nu+u_{24}\rho+k(c^2\lambda+a^2\nu+b'^2\rho)+k'=0,$$
$$u_{13}\lambda+u_{23}\mu+u_{33}\nu+u_{34}\rho+k(b^2\lambda+a^2\mu+c'^2\rho)+k'=0,$$
$$u_{14}\lambda+u_{24}\mu+u_{34}\nu+u_{44}\rho+k(a'^2\lambda+b'^2\mu+c'^2\nu)+k'=0,$$

where k, k' are arbitrary multipliers. Therefore, multiplying by λ, μ, ν, ρ, and adding, and writing ϕ for $\phi(\lambda\mu\nu\rho)$ for shortness,

$$\phi-2k=0,$$

therefore
$$\phi = 2k,$$
therefore eliminating λ, μ, ν, ρ, k' from the above four equations, and
$$\lambda+\mu+\nu+\rho = 0,$$
we have
$$\begin{vmatrix} u_{11}, & u_{12}+\frac{c^2\phi}{2}, & u_{13}+\frac{b^2\phi}{2}, & u_{14}+\frac{a'^2\phi}{2}, & 1 \\ u_{12}+\frac{c^2\phi}{2}, & u_{22}, & u_{23}+\frac{a^2\phi}{2}, & u_{24}+\frac{b'^2\phi}{2}, & 1 \\ u_{13}+\frac{b^2\phi}{2}, & u_{23}+\frac{a^2\phi}{2}, & u_{33}, & u_{34}+\frac{c'^2\phi}{2}, & 1 \\ u_{14}+\frac{a'^2\phi}{2} & u_{24}+\frac{b'^2\phi}{2}, & u_{34}+\frac{c'^2\phi}{2}, & u_{44}, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0 \ldots\ldots (6),$$
a cubic which gives three values for ϕ, and then (4) gives the squares of the semi-axes.

To find $\phi(\alpha_0\beta_0\gamma_0\delta_0)$, let
$$2k = \frac{d\phi}{d\alpha_0} = \frac{d\phi}{d\beta_0} = \frac{d\phi}{d\gamma_0} = \frac{d\phi}{d\delta_0};$$
then
$$2\phi(\alpha_0\beta_0\gamma_0\delta_0) = \alpha_0\frac{d\phi}{d\alpha_0}+\beta_0\frac{d\phi}{d\beta_0}+\gamma_0\frac{d\phi}{d\gamma_0}+\delta_0\frac{d\phi}{d\delta_0} = 2k,$$
and
$$\begin{aligned} u_{11}\alpha_0+u_{12}\beta_0+u_{13}\gamma_0+u_{14}\delta_0-k &= 0, \\ u_{12}\alpha_0+u_{22}\beta_0+u_{23}\gamma_0+u_{24}\delta_0-k &= 0, \\ u_{13}\alpha_0+u_{23}\beta_0+u_{33}\gamma_0+u_{34}\delta_0-k &= 0, \\ u_{14}\alpha_0+u_{24}\beta_0+u_{34}\gamma_0+u_{44}\delta_0-k &= 0, \end{aligned}$$
and
$$\alpha_0+\beta_0+\gamma_0+\delta_0 = 1;$$
therefore
$$k\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & u_{14}, & 1 \\ u_{12}, & u_{22}, & u_{23}, & u_{24}, & 1 \\ u_{13}, & u_{23}, & u_{33}, & u_{34}, & 1 \\ u_{14}, & u_{24}, & u_{34}, & u_{44}, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix} = -\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & u_{14} \\ u_{12}, & u_{22}, & u_{23}, & u_{24} \\ u_{13}, & u_{23}, & u_{33}, & u_{34} \\ u_{14}, & u_{24}, & u_{34}, & u_{44} \end{vmatrix}$$
therefore
$$k = -\frac{H}{B},$$
where H is the discriminant of ϕ, B the discriminant bordered with 1's.

4. Let ϕ_1, ϕ_2, ϕ_3 be the roots of (6). We have
$$\phi_1\phi_2\phi_3 = -\frac{4B}{\Sigma},$$
where
$$\begin{aligned} -\Sigma = {} & a^4a'^2+b^4b'^2+c^4c'^2+a^2a'^4+b^2b'^4+c^2c'^4 \\ & -a^2b^2c^2-a^2b^2c'^2-a^2b'^2c^2-a^2b'^2c'^2-a'^2b^2c^2-a'^2b^2c'^2 \\ & -a'^2b'^2c^2-a'^2b'^2c'^2-\&c. = -144V^2, \end{aligned}$$

where V is the volume of the fundamental tetrahedron; therefore, if r_1^2, r_2^2, r_3^2 are the semi-axes of the quadric,

$$r_1^2\, r_2^2\, r_3^2 = -\frac{k^3}{\phi_1\phi_2\phi_3} = \frac{-\frac{H^3}{B^3}}{\frac{4B}{\Sigma}},$$

therefore the volume of the quadric, if an ellipsoid,

$$= \tfrac{4}{3}\varpi r_1 r_2 r_3 = \varpi\frac{(-H)^{\frac{3}{2}}\sqrt{\Sigma}}{B^2}\cdot\frac{2}{3}$$

$$= 8\varpi V.\frac{(-H)^{\frac{3}{2}}}{B^2}.$$

5. To find the axes of any section of the quadric by a plane through the centre.

We have to make $\phi\,(\lambda\mu\nu\rho)$ a maximum or a minimum subject to the conditions (2) and (3), together with

$$l\lambda+m\mu+n\nu+p\rho=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

if $$l\alpha+m\beta+n\gamma+p\delta=0$$

be the equation to the central section.

Differentiating, we obtain equations (5) together with

$$l\delta\lambda+m\delta\mu+n\delta\nu+p\delta\rho=0 \ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Multiplying the last two equations of (5) and (8) by k_1, k_2, k_3, respectively, adding and equating to zero the coefficients of $\delta\lambda$, $\delta\mu$, $\delta\nu$, $\delta\rho$, we have $u_{11}\lambda+u_{12}\mu+u_{13}\nu+u_{14}\rho+k_1\,(c^2\mu+b^2\nu+a'^2\rho)+k_2+lk_3=0,$

&c. $=0,$

therefore $$\phi-2k_1=0;$$

therefore, eliminating λ, μ, ν, ρ, k_2, k_3 from these, together with (2) and (7), we have

$$\begin{vmatrix} u_{11}, & u_{12}+\frac{c^2\phi}{2}, & u_{13}+\frac{b^2\phi}{2}, & u_{14}+\frac{a'^2\phi}{2}, & 1, & l \\ u_{12}+\frac{c^2\phi}{2}, & u_{22}, & u_{23}+\frac{a^2\phi}{2}, & u_{24}+\frac{b'^2\phi}{2}, & 1, & m \\ u_{13}+\frac{b^2\phi}{2}, & u_{23}+\frac{a^2\phi}{2}, & u_{33}, & u_{34}+\frac{c'^2\phi}{2}, & 1, & n \\ u_{14}+\frac{a'^2\phi}{2}, & u_{24}+\frac{b'^2\phi}{2}, & u_{34}+\frac{c'^2\phi}{2}, & u_{44}, & 1, & p \\ 1, & 1, & 1, & 1, & 0, & 0 \\ l, & m, & n, & p, & 0, & 0 \end{vmatrix} = 0 \ldots(9),$$

a quadratic for ϕ.

If $l = m = n = 0$, *i.e.*, if the plane of section is parallel to the face ABC of the fundamental tetrahedron, this becomes

$$\begin{vmatrix} u_{11}, & u_{12}+\frac{c^2\phi}{2}, & u_{13}+\frac{b^2\phi}{2}, & 1 \\ u_{12}+\frac{c^2\phi}{2}, & u_{22}, & u_{23}+\frac{a^2\phi}{2}, & 1 \\ u_{13}+\frac{b^2\phi}{2}, & u_{23}+\frac{a^2\phi}{2}, & u_{33}, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots(10).$$

If ϕ_1, ϕ_2 are the roots of this equation,

$$\phi_1\phi_2 = \frac{4\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & 1 \\ u_{12}, & u_{22}, & u_{23}, & 1 \\ u_{13}, & u_{23}, & u_{33}, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix}}{a^4+b^4+c^4-2a^2b^2-2a^2c^2-2b^2c^2}$$

$$= -\frac{\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & 1 \\ u_{12}, & u_{22}, & u_{23}, & 1 \\ u_{13}, & u_{23}, & u_{33}, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix}}{4\Delta^2} = -\frac{b_4}{4\Delta^2} \text{ suppose,}$$

where Δ is the area of the triangle ABC. Hence, if r_1, r_2 are the axes of the section, therefore $r_1^2\, r_2^2 = \dfrac{k^2}{\phi_1\phi_2} = \dfrac{\dfrac{H^2}{B^2}\,4\Delta^2}{-b_4}$,

and the area of the section, if an ellipse,

$$= \varpi r_1 r_2 = \frac{2\varpi H\Delta}{B\sqrt{-b_4}}.$$

6. To find the cone touching the quadric along anyplane section.

Let $$l\alpha+m\beta+n\gamma+p\delta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(11),$$

be the equation to the plane section.

Any quadric touching the given quadric along this section is

$$\phi(\alpha\beta\gamma\delta)+k(l\alpha+m\beta+n\gamma+p\delta)^2 = 0.$$

If this is a cone,

$$\begin{vmatrix} u_{11}+kl^2, & u_{12}+klm, & u_{13}+kln, & u_{14}+klp \\ u_{12}+klm, & u_{22}+km^2, & u_{23}+kmn, & u_{24}+kmp \\ u_{13}+kln, & u_{23}+kmn, & u_{33}+kn^2, & u_{34}+knp \\ u_{14}+klp, & u_{23}+kmp, & u_{33}+knp, & u_{44}+kp^2 \end{vmatrix} = 0,$$

or $$-H+kH' = 0,$$

where H' is H bordered with l, m, n, p; therefore the equation to the cone is $\phi(a\beta\gamma\delta)+\frac{H}{H'}(la+m\beta+n\gamma+p\delta)^2=0$ .....(12).

If the plane is the plane at infinity, we have as the equation to the asymptotic cone $\phi(a\beta\gamma\delta)+\frac{H}{B}(a+\beta+\gamma+\delta)^2=0$.

7. To find the condition that a straight line should be perpendicular to a plane.

Let $la+m\beta+n\gamma+p\delta=0$ .....(13)

be the equation to the plane, and

$$\frac{a-a_0}{\lambda}=\frac{\beta-\beta_0}{\mu}=\frac{\gamma-\gamma_0}{\nu}=\frac{\delta-\delta_0}{\rho}=r$$

be the equations to the line.

Then the plane (13) is perpendicular to the two planes

$$\mu a-\lambda\beta=(\mu a_0-\lambda\beta_0)(a+\beta+\gamma+\delta),$$

and $$\rho\gamma-\nu\delta=(\rho\gamma_0-\nu\delta_0)(a+\beta+\gamma+\delta),$$

or to the parallel planes $\mu a-\lambda\beta=0$ .....(14),

$$\rho\gamma-\nu\delta=0 \quad \text{.....(15).}$$

But the condition that (13) should be perpendicular to (14) is

$$\mu l\,.\,A^2-\lambda m\,.\,B^2-(\mu m-\lambda l)\,AB\cos\widehat{AB}$$
$$-\mu n\,AC\cos\widehat{AC}-\mu p\,AD\cos\widehat{AD}+\lambda n\,BC\cos\widehat{BC}+\lambda p\,BD\cos\widehat{BD}=0,$$

where A, B, C, D represent the areas of the four faces of the tetrahedron, and $\widehat{AB}$ is the angle between the faces A and B, and so on; therefore

$$\frac{\lambda}{lA^2-mAB\cos\widehat{AB}-nAC\cos\widehat{AC}-pAD\cos\widehat{AD}}$$
$$=\frac{\mu}{mB^2-lAB\cos\widehat{AB}-nBC\cos\widehat{BC}-pBD\cos\widehat{BD}},$$

or

$$\frac{\lambda}{A(lA-mB\cos\widehat{AB}-nC\cos\widehat{AC}-pD\cos\widehat{AD})}$$
$$=\frac{\mu}{B(-lA\cos\widehat{AB}+mB-nC\cos\widehat{BC}-pD\cos\widehat{DB})}$$
$$=\frac{\nu}{C(-lA\cos\widehat{AC}-mB\cos\widehat{BC}+nC-pD\cos\widehat{CD})}$$
$$=\frac{\rho}{D(-lA\cos\widehat{AD}-mB\cos\widehat{DB}-nC\cos\widehat{CD}+pD)}$$

by symmetry.

8. Now, let $\alpha_0\beta_0\gamma_0\delta_0$ be the centre of the quadric $\phi(\alpha\beta\gamma\delta)=0$, and the line λ, μ, ν, ρ one axis.

The tangent plane at the vertex is

$$(\alpha_0+\lambda r)\frac{d\phi}{d\alpha}+(\beta_0+\mu r)\frac{d\phi}{d\beta}+(\gamma_0+\nu r)\frac{d\phi}{d\gamma}+(\delta_0+pr)\frac{d\phi}{d\delta}=0,$$

or $\alpha(u_{11}.\overline{\alpha_0+\lambda r}+u_{12}.\overline{\beta_0+\mu r}+u_{13}.\overline{\gamma_0+\nu r}+u_{14}.\overline{\delta_0+pr})+\beta(\ldots)+\ldots=0,$

therefore $\alpha\{k+r(\lambda u_{11}+\mu u_{12}+\nu u_{13}+pu_{14})\}+\beta\{\ldots\}+\ldots=0;$

therefore, since this is perpendicular to the axis,

$$\frac{\lambda}{Ar\{\lambda(u_{11}A-u_{12}B\cos\widehat{AB}-u_{13}C\cos\widehat{AC}-u_{14}D\cos\widehat{AD})+\mu(\,)+\ldots\}}$$

$$=\frac{\mu}{\ldots}=\frac{\nu}{\ldots}=\frac{\rho}{\ldots}=\sigma \text{ suppose } \ldots\ldots\ldots\ldots\ldots\ldots(16),$$

since $$A-B\cos\widehat{AB}-C\cos\widehat{AC}-D\cos\widehat{AD}=0,$$
$$\&c. \qquad =0.$$

Adding the numerators and denominators, we see that these equations satisfy the equation $\lambda+\mu+\nu+\rho=0$ identically, as they ought.

Eliminating λ, μ, ν, ρ, we have a cubic for $r\sigma$, and, using any one of the roots of this cubic in any three of these equations, we find the values of $\lambda:\mu:\nu:\rho$ for that axis.

In order to separate r and σ, we must express that the point $\alpha_0+\lambda r$, $\beta_0+\mu r$, $\gamma_0+\nu r$, δ_0+pr lies on the surface.

9. In the case of the quadric

$$u_{11}\alpha^2+u_{22}\beta^2+u_{33}\gamma^2+u_{44}\delta^2=0,$$

with respect to which the fundamental tetrahedron is self-conjugate,

we have
$$B=\begin{vmatrix} u_{11} & 0 & 0 & 0 & 1\\ 0 & u_{22} & 0 & 0 & 1\\ 0 & 0 & u_{33} & 0 & 1\\ 0 & 0 & 0 & u_{44} & 1\\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}$$

$$=-u_{11}u_{22}u_{33}-u_{11}u_{22}u_{44}-u_{11}u_{33}u_{44}-u_{22}u_{33}u_{44},$$

$$H=\begin{vmatrix} u_{11} & 0 & 0 & 0\\ 0 & u_{22} & 0 & 0\\ 0 & 0 & u_{33} & 0\\ 0 & 0 & 0 & u_{44} \end{vmatrix}=u_{11}u_{22}u_{33}u_{44},$$

therefore the volume of the quadric, if an ellipsoid,

$$= 8\varpi V \frac{(-u_{11}u_{22}u_{33}u_{44})^{\frac{3}{2}}}{(u_{11}u_{22}u_{33}+\ldots)^2};$$

the equations giving the centre are

$$u_{11}a = u_{22}\beta = u_{33}\gamma = u_{44}\delta,$$

therefore the volume of the quadric

$$= 8\varpi V \frac{\left(-\frac{1}{a\beta\gamma\delta}\right)^3}{\left(\frac{1}{a\beta\gamma}+\ldots\right)^2}$$

$$= 8\varpi V \frac{\sqrt{-a\beta\gamma\delta}}{(a+\beta+\gamma+\delta)^2}.$$

Hence the volume of the ellipsoid varies as the square root of the product of the four perpendiculars from the centre on the faces of the fundamental tetrahedron, and, if the volume of the ellipsoid is given, the locus of the centre is

$$a\beta\gamma\delta = \lambda(a+\beta+\gamma+\delta)^4,$$

where λ is a constant.

Proofs of the Addition Theorem for Elliptic Integrals of the Second Kind, and of Fagnani's Theorem. By J. J. Walker.

[*Read March 9th*, 1882.]

In Crelle's *Journal*, Bd. xc., Herr Græfe has shown how proofs of the addition-theorems for the second and third kinds of elliptic integrals, analogous to Lagrange's proof for the first kind, may be obtained from the properties of the spherical triangle.

It seems natural to look to the ellipse as a means of proof for the second kind; and a property of tangents drawn from an external point to which I have recently been led, combined with the theorem relating to the excess of the sum of two such tangents over the intercepted arc, affords such a proof; it also conducts, by taking a particular case, to a very simple independent demonstration of Fagnani's well-known theorem.

The property referred to is this:—If from a point on an ellipse tan-

gents be drawn to an inner confocal ellipse, their sum $(t+t')$ is equal to

$$\frac{2a'b'}{k}\sin^2\tfrac{1}{2}(\theta-\theta') \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where a', b' are the semi-axes of the external ellipse, and

$$k^2 = a'^2-a^2 = b'^2-b^2,$$

a, b being the semi-axes of the interior ellipse; while θ, θ' are the eccentric angles, or angular parameters, of the points of contact.

It is easily proved, and is well-known, that if from a point on an ellipse, with semi-axes a', b', tangents are drawn to the interior ellipse with semi-axes a, b, then the angular parameters of the points of contact are connected by the relation

$$\frac{a^2}{a'^2}\cos^2\frac{\theta+\theta'}{2}+\frac{b^2}{b'^2}\sin^2\frac{\theta+\theta'}{2}=\cos^2\frac{\theta-\theta'}{2},$$

viz.,

$$\left(1-\frac{a^2}{a'^2}+\frac{b^2}{b'^2}\right)\cos\theta\cos\theta'+\left(1+\frac{a^2}{a'^2}-\frac{b^2}{b'^2}\right)\sin\theta\sin\theta'=-1+\frac{a^2}{a'^2}+\frac{b^2}{b'^2},$$

or, writing $\qquad \theta=\frac{\pi}{2}-\phi, \quad \theta'=\frac{\pi}{2}\pm\psi,$

$$\left(1+\frac{a^2}{a'^2}-\frac{b^2}{b'^2}\right)\cos\phi\cos\psi\mp\left(1-\frac{a^2}{a}+\frac{b^2}{b'^2}\right)\sin\phi\sin\psi=-1+\frac{a^2}{a'^2}+\frac{b^2}{b'^2}$$

$\ldots\ldots\ldots\ldots(2).$

Identifying this relation with

$$\cos\phi\cos\psi\mp\sqrt{1-k^2\sin^2\mu}\,\sin\phi\sin\psi=\cos\mu,$$

$$\cos\mu=\left(-1+\frac{a^2}{a'^2}+\frac{b^2}{b'^2}\right)\div D;$$

whence $\qquad \sin\mu=\frac{2ak}{a'b'}\div D,$

$$\kappa^2=(a^2-b^2)\div a^2=e^2.$$

Now, by the theorem commonly known as Graves' Theorem, the sum of the tangents drawn from any external point to an ellipse exceeds the length of the intercepted arc by a constant quantity, or by (1),

$$u\pm v=\frac{2a'b'}{ak}\sin^2\tfrac{1}{2}(\theta-\theta')+\text{constant}$$

$$=\frac{2a'b'}{ak}\sin^2\tfrac{1}{2}(\phi\pm\psi)+C$$

$$=\frac{a'b'}{ak}(1-\cos\phi\cos\psi\pm\sin\phi\sin\psi)+C,$$

where $u = \int_0^{\phi} \sqrt{1-e^2\sin^2\phi}\, d\phi, \quad v = \int_0^{\psi} \sqrt{1-e^2\sin^2\psi}\, d\psi.$

Eliminating $\cos\phi\cos\psi$ by means of (2), there results

$$u \pm v = e^2 \sin\mu \sin\phi \sin\psi + C';$$

which is the addition theorem for the elliptic integrals of the second kind.

Consider now the confocal ellipse, the squares of whose semi-axes are $a'^2 = a(a+b), \quad b'^2 = b(a+b),$

giving $a'^2 - b'^2 = a^2 - b^2;$

for these values of a'^2, b'^2,

$$1 - \frac{a^2}{a'^2} + \frac{b^2}{b'^2} = \frac{2b}{a+b}, \quad 1 + \frac{a^2}{a'^2} - \frac{b^2}{b'^2} = \frac{2a}{a+b},$$

$$1 - \frac{a^2}{a'^2} - \frac{b^2}{b'^2} = 0, \quad \frac{a'b'}{k} = a+b;$$

so that the relation (2) between θ and θ' becomes

$$b\cos\theta\cos\theta' + a\sin\theta\sin\theta' = 0,$$

or $$\tan\theta\tan\theta' = -\frac{b}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

viz., if T, T' are the points of contact, these are "Fagnani Points."

The outer ellipse, in this case, is obviously that passing through the corners of the rectangle circumscribing the inner; so that, calling the quadrantal arc of this AB, and the arc intercepted between the points T, T', TT',

$$TT' - (t+t') = AB - (a+b),$$

or
$$\begin{aligned} AT - BT' &= (a+b)\left(1 - 2\sin^2\frac{\theta-\theta'}{2}\right) \\ &= (a+b)(\cos\theta\cos\theta' + \sin\theta\sin\theta') \\ &= -\frac{a^2-b^2}{b}\sin\theta\sin\theta', \end{aligned}$$

or $$BT' - AT = TY = T'Y',$$

Y, Y' being the feet of the perpendiculars on the tangents at TT' from the centre. This is Euler's form of Fagnani's Theorem.

The general value of TY is $\dfrac{(a^2-b^2)\sin\theta\cos\theta}{\sqrt{(a^2\sin^2\theta + b^2\cos^2\theta)}}$; but, in consequence of the relation (3),

$$\tan\theta' = -\frac{b\cos\theta}{a\sin\theta}, \quad \frac{\sin\theta'}{b} = \frac{\cos\theta}{\sqrt{(a^2\sin^2\theta + b^2\cos^2\theta)}},$$

and therefore $$\frac{(a^2-b^2)\sin\theta\cos\theta}{\sqrt{(a^2\sin^2\theta + b^2\cos^2\theta)}} = \frac{a^2-b^2}{b}\sin\theta\sin\theta'.$$

Thursday, May 11th, 1882.

S. ROBERTS, Esq., F.R.S., President, in the Chair.

Mr. A. L. Daniels, Phillipp Strasse, Berlin, was elected a member. Dr. Hirst spoke upon a method of General Inversion, due to M. Vanecek (of Prague).

The following papers were received and partly communicated by the Secretary:—

"On a Formula relating to Elliptic Integrals of the Third Kind:" Prof. Cayley.

"Elementary Analytical Proof of Graves's and MacCullagh's Theorems, with an extension of the former:" J. Griffiths, M.A.

"Note on a System of Confocal Bicircular Quartics:" R. A. Roberts, M.A.

"On the Vibrations of an Elastic Sphere:" Prof. H. Lamb.

The President also made a short communication.

The following presents were received:—

"Educational Times," May, 1882.

"Bulletin de la Société Mathématique de France," Tome x., No. 2.

"Atti della R. Accademia dei Lincei—Transunti," Vol. vi., Fasc. 9°, 10°; Roma.

"Journal of Institute of Actuaries," No. cxxv., Vol. xxiii., Pt. iii.

"Bulletin des Sciences Mathématiques et Astronomiques," Tome v., Oct. 1881.

"Proceedings of Royal Society," Vol. xxxiii., No. 218.

"Proceedings of the Royal Society of Edinburgh," Session 1880-1, pp. 1-295.

"Beiblätter zu den Annalen der Physik und Chemie," Band vi., Stück 4, No. 4, 1882.

"Bulletin Astronomique et Météorologique de l'Observatoire Impérial de Rio de Janeiro," Oct., Nov., Déc., 1881, Nos. 4, 5, 6; Rio de Janeiro, 1881.

"Aggiunte a recenti lavori dei Sig. Weierstrass e Mittag-Leffler sulle funzioni di una Variabile Complessa di F. Casorati."

"Ueber eindeutige Functionen mitlinearen Substitutionen in sich (Zweite Mittheilung)," von F. Klein (Math. Ann. xx., pp. 49—51).

"On Riccati's Equation and its Transformations, and on some Definite Integrals which satisfy them," by J. W. L. Glaisher, F.R.S. (Phil. Trans., Pt. iii., 1881): from the Author.

On a Formula relating to Elliptic Integrals of the Third Kind.

By A. CAYLEY.

[*Presented May 11th*, 1882.]

The formula for the differentiation of the integral of the third kind $\Pi = \int_0 \frac{d\phi}{(1+n\sin^2\phi)\,\Delta}$ in regard to the parameter n, see my "Elliptic Functions," Nos. 174 *et seq.*, may be presented under a very elegant

form, by writing therein $\sin^2\phi = x = \mathrm{sn}^2 u$, $\sin\phi\cos\phi\,\Delta = y = \mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u$, and thus connecting the formula with the cubic curve

$$y^2 = x(1-x)(1-k^2x).$$

The parameter must, of course, be put under a corresponding form, say $n = -\frac{1}{a}$, where $a = \mathrm{sn}^2\theta$, $b = \mathrm{sn}\,\theta\,\mathrm{cn}\,\theta\,\mathrm{dn}\,\theta$, and therefore (a, b) are the coordinates of the point corresponding to the argument θ. The steps of the substitution may be effected without difficulty, but it will be convenient to give at once the final result and then verify it directly. The result is

$$\frac{d}{d\theta}\frac{b}{a-x} - \frac{d}{du}\frac{y}{x-a} = k^2(a-x).$$

We, in fact, have $\frac{dx}{du} = 2\,\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u = 2y$,

and thence $$y\frac{dx}{du} = 2y^2,$$

that is $$y\frac{dx}{du} = 2x\left[1-(1+k^2)x+k^2x^2\right].$$

Also $$\frac{dy}{du} = \mathrm{cn}^2 u\,\mathrm{dn}^2 u - \mathrm{sn}^2 u\,\mathrm{dn}^2 u - k^2\,\mathrm{sn}^2 u\,\mathrm{cn}^2 u$$
$$= 1-2(1+k^2)x+3k^2x^2,$$

and hence

$$\frac{d}{du}\frac{y}{x-a} = \frac{1}{(a-x)^2}\left\{(x-a)\frac{dy}{du} - y\frac{dx}{du}\right\}$$
$$= \frac{1}{(a-x)^2}\left\{-x-a+2(1+k^2)ax+k^2x^3-3k^2x^2\right\}.$$

Interchanging the letters, we have

$$\frac{d}{d\theta}\frac{b}{a-x} = \frac{1}{(a-x)^2}\left\{-x-a+2(1+k^2)ax+k^2a^3-3k^2a^2x\right\},$$

and hence, subtracting,

$$\frac{d}{d\theta}\frac{b}{a-x} - \frac{d}{du}\frac{y}{x-a} = \frac{1}{(a-x)^2}\left\{k^2a^3-3k^2a^2x+3k^2ax^2-k^2x^3\right\}$$
$$= \frac{1}{(a-x)^2}k^2(a-x)^3$$
$$= k^2(a-x),$$

which is the required result.

Elementary Analytical Proof of Graves's and MacCullagh's Theorems, with an extension of the former. By JOHN GRIFFITHS, M.A.

[*Read May 11th*, 1882.]

The elementary proof in question to which I was led some time ago depends upon the following proposition, viz.,

If $$t^2 = (a\sin\theta - a'\sin\theta')^2 + (b\cos\theta - b'\cos\theta')^2,$$

where $\frac{a}{a'}\sin\theta\sin\theta' + \frac{b}{b'}\cos\theta\cos\theta' = 1$, and $a^2 - b^2 = a'^2 - b'^2$,

then $$dt = \sqrt{a'^2\cos^2\theta + b'^2\sin^2\theta}\, d\theta - \sqrt{a'^2\cos^2\theta' + b'^2\sin^2\theta'}\, d\theta'.$$

To prove this, let us write

$$a\sin\theta - a'\sin\theta' = x, \quad b\cos\theta - b'\cos\theta' = y,$$

$$\frac{a}{a'}\sin\theta\sin\theta' = \lambda, \quad \frac{b}{b'}\cos\theta\cos\theta' = 1-\lambda, \quad \lambda - \sin^2\theta' = \lambda',$$

$$\sqrt{a'^2\cos^2\theta + b'^2\sin^2\theta} = \Delta\theta, \quad \sqrt{a'^2\cos^2\theta' + b'^2\sin^2\theta'} = \Delta\theta';$$

so that $$x = \frac{\lambda' a'}{\sin\theta'}, \quad y = -\frac{\lambda' b'}{\cos\theta'}, \quad t = \sqrt{x^2+y^2} = \frac{\lambda'.\Delta\theta'}{\sin\theta'\cos\theta'},$$

$$ax\cos\theta - by\sin\theta = \frac{\lambda'(aa'\cos\theta\cos\theta' + bb'\sin\theta\sin\theta')}{\sin\theta'\cos\theta'},$$

$$= \frac{t(aa'\cos\theta\cos\theta' + bb'\sin\theta\sin\theta')}{\Delta\theta'},$$

and, similarly, $$a'x\cos\theta' - b'y\sin\theta' = t.\Delta\theta'.$$

Now, differentiating the expression

$$t^2 = (a\sin\theta - a'\sin\theta')^2 + (b\cos\theta - b'\cos\theta')^2,$$

we have $$t\,dt = (ax\cos\theta - by\sin\theta)\,d\theta - (a'x\cos\theta' - b'y\sin\theta')\,d\theta'$$

$$= \frac{t(aa'\cos\theta\cos\theta' + bb'\sin\theta\sin\theta')}{\Delta\theta'}\,d\theta - t.\Delta\theta' d\theta',$$

or $$dt = \Delta\theta.\,d\theta - \Delta\theta'.\,d\theta',$$

if $$aa'\cos\theta\cos\theta' + bb'\sin\theta\sin\theta' = \Delta\theta.\Delta\theta'.$$

To verify that

$$(aa'\cos\theta\cos\theta' + bb'\sin\theta\sin\theta')^2 = (a'^2\cos^2\theta + b'^2\sin^2\theta)(a'^2\cos^2\theta' + b'^2\sin^2\theta')$$

is an identity, subject to the above conditions, we may divide both sides

of the equation in question by $(a'b')^2$, so that

$$\left\{\frac{a}{b}(1-\lambda)+\frac{b}{a}\lambda\right\}^2=\frac{a'^2}{b^2}(1-\lambda)^2+\frac{b'^2}{a^2}\lambda^2+\sin^2\theta'\cos^2\theta+\sin^2\theta\cos^2\theta',$$

or, since
$$\cos^2\theta-\cos^2\theta\sin^2\theta'=\frac{b'^2}{b^2}(1-\lambda)^2,$$

$$\sin^2\theta-\sin^2\theta\cos^2\theta'=\frac{a'^2}{a^2}\lambda^2,$$

and, consequently,

$$\cos^2\theta\sin^2\theta'+\cos^2\theta'\sin^2\theta=1-\frac{a'^2}{a^2}\lambda^2-\frac{b'^2}{b^2}(1-\lambda)^2;$$

this may be written

$$\left\{\frac{a}{b}(1-\lambda)+\frac{b}{a}\lambda\right\}^2=\frac{a'^2}{b^2}(1-\lambda)^2+\frac{b'^2}{a^2}\lambda^2+1-\frac{a'^2}{a^2}\lambda^2-\frac{b'^2}{b^2}(1-\lambda)^2,$$

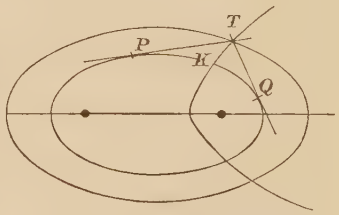
which is $\dfrac{a^2-a'^2+b'^2}{b^2}(1-\lambda)^2+\dfrac{b^2-b'^2+a'^2}{a^2}\lambda^2+2\lambda(1-\lambda)=1,$

i.e., finally, $(1-\lambda)^2+\lambda^2+2\lambda(1-\lambda)=1$; if $a^2-b^2=a'^2-b'^2$.

The geometrical interpretation of the result

$$dt=\Delta\theta\,.\,d\theta-\Delta\theta'\,.\,d\theta',$$

just established, is as follows :—



Let TP, TQ be the tangents to a given ellipse PKQ, of semi-axes a', b', from a point T on another confocal ellipse of semi-axes a, b; K the point where the confocal hyperbola TK drawn through T cuts the inner ellipse; then the above theorem asserts that arc PK—tangent TP is a constant quantity for the two ellipses.

This is, therefore, the extension of Graves's theorem which we are led to.

Again, writing $TP-\text{arc}\,KP=C$ and $TQ-\text{arc}\,KQ=C'$, it is easily seen, by taking T at a special position on the outer ellipse, *i.e.*, at an extremity of the minor axis, for instance, that $C'=C$.

Hence Graves's theorem follows by addition and MacCullagh's by subtraction.

The above results give the addition-theorem for elliptic integrals of the second kind in a slightly different form from the usual one; but the identity of the two forms may be established by showing, as is

easily done, that if

$$\frac{a}{a'}\sin\theta\sin\theta' + \frac{b}{b'}\cos\theta\cos\theta' = 1, \quad a^2 - b^2 = c^2 = a'^2 - b'^2,$$

$a^2 - a'^2 = c'^2 = b^2 - b'^2$, and $t^2 = (a\sin\theta - a'\sin\theta')^2 + (b\cos\theta - b'\cos\theta')^2$,

then

$$\frac{a^2}{c^2} \mp \frac{ab}{c'c^2}t = \frac{a}{a'}\sin\theta\sin\theta',$$

$$-\frac{b^2}{c^2} \pm \frac{ab}{c'c^2}t = \frac{b}{b'}\cos\theta\cos\theta'.$$

I may also observe that the formulæ can be reduced to the standard forms by writing $a' = 1$, $b' = b\cos\mu$, and, consequently,

$$a = \frac{\sqrt{1 - c^2\sin^2\mu}}{\cos\mu}; \quad c' = b\sin\mu.$$

Thursday, June 8th, 1882.

S. ROBERTS, Esq., F.R.S., President, in the Chair.

Messrs. J. W. Berry, Pittston, Pa.; A. R. Forsyth, B.A., Fellow of Trinity College, Cambridge; and James Wood, M.A., Fellow of Queens' College, Cambridge, were elected Members; and Mr. R. A. Roberts, M.A., was admitted into the Society.

The following communications were made:—

"On the Extension of certain Theories relating to Plane Cubics to Curves of any Deficiency:" by A. Buchheim.

"On the Differentiation with respect to the Modulus of the Amplitude of Elliptic Functions:" by Rev. M. M. U. Wilkinson.

"Two Notes: (1) A Definite Integral; (2) Equation to the Director-Circle of a Conic:" by Prof. Wolstenholme.

"Theory of Orthoptic Loci:" by Rev. Dr. Taylor.

The President, Revs. R. Harley and Wilkinson, and Mr. J. Hammond took part in discussions on the papers.

The following presents were received:—

Cabinet likeness of Prof. Cremona for the Album.

"Cyklographie, oder Construction der Aufgaben über Kreise und Kugeln, und Elementare Geometrie der Kreis- und Kugel-Systeme," von Dr. Wilhelm Fiedler (mit 16 Lithog. Tafeln); Leipzig, 1882. [The above presents were made, on behalf of the donors, by Dr. Hirst.]

Cartes-de-Visite likenesses were received from Revs. A. Freeman, J. F. Twisden, and M. M. U. Wilkinson.

"The Mathematical Magazine, a Journal of Elementary Mathematics," edited and published by Artemas Martin, M.A., Vol. i., No. 2, April, 1882.
"The Pasturage Problem," by A. Martin, M.A. (Extract from above magazine.)
"Educational Times," June, 1882.
"Proceedings of Royal Society," Vol. xxiii., No. 219.
"Bulletin des Sciences Mathématiques et Astronomiques," Tome v., Nov.–Dec., 1881.
"Bulletin de la Société Mathématique de France," Tome x., No. 3.
"Atti della R. Accademia dei Lincei," Vol. vi., Fasc. 12° (in duplicate).
"Beiblatter zu den Annalen der Physik und Chemie," Band vi., Stück v., No. 5, 1882.
"Oversigt over det Kongelige Danske Videnskabernes Selskabs Forhandlinger og dets Medlemmers Arbejder i Aaret 1881," Kjobenhavn, No. 3, 1881: presented by Sir J. Lubbock, F.R.S.
"Washington Observations for 1877"—Appendix ii., "The Multiple Star Σ 748," by Ed. S. Holden; Washington, 1881.
"Annali di Matematica Pura ed Applicata," Serie ii^a, Tomo x°, Fasc. 4° (Maggio, 1882); Milano.
"American Journal of Mathematics," Vol. iv., No. 3; Baltimore (Sept. 1881), 1882.

On the Extension of certain Theories relating to Plane Cubics to Curves of any Deficiency. *By* Arthur Buchheim, B.A.

[*Read June 8th*, 1882.]

The object of the following paper is the extension, by the use of Abelian functions, of certain theories which, in the case of plane cubics, are immediate consequences of the representation of the coordinates of a point of the curve as elliptic functions of a parameter. The theories considered are: (i.) The theory of Steiner's Polygons, and (ii.) Prof. Sylvester's theory of Derived Points. The curves considered are curves of order n and deficiency p in an $(n-p)$ flat.

I write k for $n-p$, and m, r respectively for the quotient and remainder of the division of k by p, so that $k = mp+r$ and $r<p$; we have also $n = (m+1)p+r$.

Every point on the curve has p parameters, viz., the values for that point of the p Abelian integrals of the first species appertaining to the curve; in the equations that follow, only one parameter is written down, so that every equation stands for p equations. We have to consider groups of p points, and sets of groups of $(m-1)$ groups each; each set therefore containing $(m-1)p$ points. A group of p points will generally be called simply a *group*, and a set of $(m-1)p$ points will generally be called simply a *set*.

*Steiner's Polygons.**

Take two sets, of $(m-1)p$ points each, on the curve: call the sum of their parameters a and b respectively; take a group of p points, and let the sum of the parameters be u; lastly, take a group of r points, and let the sum of their parameters be c.† Draw a $(k-1)$ flat through u, passing through c and a: this cuts the curve in a group v_1, containing p points: draw a $(k-1)$ flat through v_1, c and b cutting in a group v_2; draw a $(k-1)$ flat through v_2, c and a cutting in a group v_3, and so on. We thus get a configuration which we may call a polygon, each vertex of the "polygon" being a group of p points: the polygon will be closed if the last group coincides with the first. We have the congruences

$$u+a+c+v_1 \equiv 0,$$
$$v_1+b+c+v_2 \equiv 0,$$
$$v_2+a+c+v_3 \equiv 0,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

It follows from these equations that for a "closed $2n$-gon" we must have $$n(a-b) \equiv 0;$$

the theory of these polygons is thus seen to be entirely analogous to the theory of Steiner's polygons. I shall now show how Steiner's theorems‡ can be extended to the curves considered in this paper.

One of Steiner's theorems is the following:—"Let P, Q be two points, R their residual, S the point of contact of a tangent from R; then, if taking P, Q as base points, we get a polygon of $2n$ sides, we shall get a polygon of $4n$ sides if we take P and S as base points." The corresponding theorem is as follows:—"Let P and Q be two sets [of $(m-1)p$ points], T a fixed group of r points: divide the two sets into groups, P_i, Q_i, and establish a correspondence between the groups. Draw a $(k-1)$ flat having $(m-1)$-point contact in every point of P_i and passing through Q_i and T: let this cut in a group R_i: through R_i, T draw a $(k-1)$ flat, having m-point contact in a group S_i. We thus get $(m-1)$ groups S_i forming a set S; then, if P, Q give a $2n$-gon, P, S will give a $2mn$-gon. The proof is very simple; we have

$$(m-1)P_i+Q_i+T+R_i \equiv 0,$$
$$T+R_i+mS_i \equiv 0.$$

Therefore
$$(m-1)P_i+Q_i-mS_i \equiv 0,$$
$$Q_i \equiv mS_i-(m-1)P_i,$$

* In his paper "Ueber die Anwendung der Abel'schen Functionen in der Geometrie," Clebsch gives a very general theorem, extending the theory of Steiner's polygons to curves in three-dimensional space which are the complete intersections of surfaces.

† A group or set for which the sum of the parameters is v, will be referred to as the group or set v.

‡ Steiner, "Geometrische Lehrsätze," *Crelle*, t. xxxii., p. 182; Ges. Werke, t. ii., p. 371.

$$Q \equiv mS - (m-1)P,$$

$$P - Q \equiv m(P - S).$$

But $$n(P - Q) \equiv 0.$$

Therefore $$nm(P - S) \equiv 0,$$

and hence the theorem follows.

It seems hardly worth while to give further examples; it is, however, worth while to remark that, if $m=2$, $r=0$, Steiner's theorems can be extended by replacing "point" by "group of p points," and straight line by $(k-1)$ flat: these are the only alterations required.*

Derived Points.

In his paper "On the Classification of Loci,"† Clifford gives an account of Prof. Sylvester's theory of derived points on a plane cubic, and remarks that the theory is really a geometric representation of the multiplication of elliptic functions. He says:—"I was desirous of finding a similar representation of the multiplication of hyper-elliptic and Abelian functions; and therefore sought for cases in which derived elements might be found on *curves* . . . . of deficiency greater than 1. . . . . It may be shown, in general, that a *curve* on which such a theory of derived points is possible, is at most of deficiency 1. . . . Thus the impossibility of extending the theory of derivation to curves of deficiency greater than unity is equivalent to the proposition that a curve of order k in $k-1$ dimensions is at most of deficiency 1." It is, however, possible to extend the theory to curves of any deficiency, if, instead of seeking for derived *points*, we seek for derived *groups* (of p points); and, indeed, this seems the most natural extension of the theory, for the form in which the problem of the multiplication of Abelian functions is ordinarily presented‡ requires us to find a group, such that the sum of the parameters is a multiple of the sum of the parameters of a given group. If we seek for such an extension of the theory, we are at once led to the curves considered in this paper; for, suppose we have a curve (without singular points) in a k flat, such that k points being given a single group of p points is determined by drawing a $(k-1)$ flat through the k points; then, since a $(k-1)$ flat cuts a curve of order n in n points, the order of the curve must be $k+p$.

The extension of Prof. Sylvester's theory is now quite obvious. We

* The following theorem (enunciated for simplicity for the case of $m = 2$, $r = 0$, but true *mutatis mutandis* for all values) includes the rest of Steiner's theorems: "Take two groups P, Q, draw $(k-1)$ flats touching the curve in every point of P, Q, and cutting in $P_1 Q_1$; draw $(k-1)$ flats through PQ_1, $P_1 Q$, cutting in $P_2 Q_2$; derive $P_3 Q_3$ from these as they were got from $P_1 Q_1$, and so on. Then, for a $2(n+1)$-gon, $P_n Q_n$ must coincide, n being any integer."

† *Phil. Trans.*, 1878, p. 663; Mathematical Papers, p. 304.

‡ Clebsch and Gordan, *Abel'sche Functionen*, p. 230.

take a group U and r fixed points, R on the curve; we draw a $(k-1)$ flat, having m-point contact in every point of U, passing through R and cutting in a new group V_1; from V_2 we derive a new group in the same way, and so on. We thus get a series of groups, and by drawing a $(k-1)$ flat through any m groups, no two of which are consecutive, and r points, we get a new group.

We have

$$V_1 \equiv -mU - R_1,$$
$$V_2 \equiv -mV_1 - R_2 \equiv m^2U + mR_1 - R_2,$$
$$V_3 \equiv -m^3U - m^2R_1 + mR_2 - R_3,$$
$$V_n \equiv (-)^n \{m^nU + m^{n-1}R_1 - m^{n-2}R_2 + m^{n-3}R_3 \ldots\} - R_n,$$

and it is easy to see that no coefficient in any derived group is divisible by $(m+1)$.

In the case of the cubic we have the theorem—The coordinates of any derived point may be expressed rationally in terms of the coordinates of the original point, and the order of the functions to which they are proportional is always a square number. The corresponding theorem is—Given a derived group, the determination of the original group depends on an equation of order p, whose coefficients depend on an equation of which the order is always of the form n^{2p}, and the coefficients of this last equation are rational functions of the coordinates of the given group.

Application to Plane Curves.

I show, in this section, how the curves here considered may be derived from plane curves of deficiency p. Suppose we have a ∞^k-series of curves cutting a given curve C_n in m variable points; let $\psi_1 \ldots \psi_{k+1}$ be $(\kappa+1)$ asyzygetic curves of the series; then, if we take

$$x_1 : x_2 : x_3 : \&c. = \psi_1 : \psi_2 : \psi_3 : \&c.,$$

$(x_1 : x_2 : x_3 \ldots : x_\kappa)$ will determine a point on a curve in a space of k dimensions, and the order of the curve will be the number of points in which the C_n is cut by a curve $\Sigma a\psi = 0$; viz., the new curve will be of order m.

We have now to find curves ψ giving $m = k+p$. As we wish to apply Jacobi's problem of inversion to the intersections with a curve ψ, it must be a curve passing $i-1$ times through each i-tuple point of the C_n;* such curves may be called *associate*† curves. Suppose we have a series of associate curves of order λ passing through a fixed points on the curve, and transforming the curve into a C_{k+p} in a k-flat.

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* Brill and Nöther, *Math. Ann.*, t. vii., or Clebsch's *Lectures*, pp. 430, &c.

† Brill and Nöther's "Adjungirte Curven."

We must have (if the multiple points are equivalent to δ double points),

$$2\delta + p + k + a = n\lambda,$$

$$\delta + k + a = \frac{\lambda \,.\, \lambda + 3}{2},$$

therefore
$$+p = n\lambda - \frac{\lambda \,.\, \lambda + 3}{2},$$

or
$$n-1 \,.\, n-2 = 2n\lambda - \lambda \,.\, \lambda + 3;$$

this gives
$$\lambda^2 - (2n-3)\,\lambda + n-1 \,.\, n-2 = 0,$$

$$\lambda = \frac{2n-3 \pm 1}{2}$$

$$= n-1 \text{ or } n-2.$$

The first value of λ gives

$$k + a = \frac{n-1 \,.\, n+2}{2} - \delta$$

$$= \frac{n-1 \,.\, n-2}{2} - \delta + 2\,(n-1)$$

$$= p + 2\,(n-1),$$

and the second value of λ gives

$$k + a = p + n - 2.$$

Since a must be positive or zero, we must have, for $\lambda = n-1$,

$$k \leqq p + 2\,(n-1),$$

and for $\lambda = n-2$,
$$k \leqq p + n - 2,$$

we can always transform the curve into a C_{3p} in a $2p$-flat; in certain cases we can do this in one step, but it can always be done in two steps. If the transformation is to be effected in one step, we must have

$$2p \leqq p + 2\,(n-1), \text{ or } p + n - 2,$$

$$p \leqq 2\,(n-1), \text{ or } n-2,$$

This is always satisfied if n is less than 6.

To effect the transformation in two steps, we first transform the curve into a plane curve of order $p+2$. This is always possible: we have to take
$$a = p + 2n - 4 \text{ for } \lambda = n-1,$$

and
$$a = p + n - 4 \text{ for } \lambda = n-2.$$

The plane curve of order $p+2$ can be transformed into the curve of order $3p$ in a $2p$-flat, by taking

$$a = p+2,$$

and transforming by curves of order $p+1$. The transformation by curves of order p is inapplicable, as it gives a negative value for a.

Two Notes: — (1) *A Definite Integral;* (2) *Equation of the Director-Circle of a Conic.* *By* Prof. Wolstenholme.

[*Read June 8th*, 1882.]

I. If $$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2} f''(0) + \ldots + \frac{x^n}{\lfloor n} X,$$

and r be any positive quantity,

$$\int_0^\infty \frac{X}{x^r}\,dx = \frac{\Gamma(n+1)\,\Gamma(r)}{\Gamma(n+r)} \int_0^\infty \frac{f^n(x)}{x^r}\,dx.$$

In general,

$$f(x) = f(0) + xf'(0) + \ldots + \frac{x^{n-1}}{\lfloor n-1} f^{n-1}(0) + \frac{1}{\lfloor n-1} \int_0^x z^{n-1} f^n(x-z)\,dz\,;$$

hence $$X = \frac{n}{x^n} \int_0^x z^{n-1} f^n(x-z)\,dz = \frac{n}{x^n} \int_0^x (x-z)^{n-1} f^n(z)\,dz,$$

and $$\int_0^\infty \frac{X}{x^r}\,dx = n \int_0^\infty \frac{dx}{x^{n+r}} \int_0^x (x-z)^{n-1} f^n(z)\,dz,$$

which, changing the order of integration, is

$$n \int_0^\infty f^n(z)\,dz \int_z^\infty \frac{(x-z)^{n-1}}{x^{n+r}}\,dx,$$

and $$\int_z^\infty \frac{(x-z)^{n-1}}{x^{n+r}}\,dx = \frac{1}{z^r} \int_1^\infty \frac{(x-1)^{n-1}}{x^{n+r}}\,dx = \frac{1}{z^r} \int_0^1 \left(\frac{1}{x} - 1\right)^{n-1} . x^{n+r} \frac{dx}{x^2}$$

$$= \frac{1}{z^r} \int_0^1 x^{r-1} (1-x)^{n-1}\,dx = \frac{1}{z^r} \frac{\Gamma(r)\,\Gamma(n)}{\Gamma(n+r)},$$

if r be any positive quantity, but is ∞ if r be zero or negative.

We have then

$$\int_0^\infty \frac{X}{x^r}\,dx = n \frac{\Gamma(r)\,\Gamma(n)}{\Gamma(n+r)} \int_0^\infty \frac{f^n(z)}{z^r}\,dz = \frac{\Gamma(n+1)\,\Gamma(r)}{\Gamma(n+r)} \int_0^\infty \frac{f^n(x)}{x^r}\,dx.$$

The integrals will generally be finite, if r be a proper fraction.

A particular case of this [$f(x) = \sin x$ or $\cos x$, $r = 1$] was set in the Cambridge Senate House Examination in 1875, and was discovered by me by integrating by parts $\frac{\sin x}{x}$ any number of times, always making the integrated part vanish with x. This method would give the general theorem apparently only for functions of the form ϵ^{ax}. The proof given in the published solutions of the Senate House questions for 1875, p. 132, is altogether erroneous: writing

$$\sin x = x - \frac{x^3}{\lfloor 3} + \ldots\ldots + (-1)^n X \frac{x^{2n}}{\lfloor 2n},$$

it asserts correctly that $X = \sin \theta x$ where θ is a positive proper fraction, but erroneously goes on to say that therefore

$$\int_0^\infty \frac{\sin \theta x}{x}\, dx = \frac{\pi}{2}$$

overlooking the fact that θ is a function of x.

If we put $f(x) = \sin x$, $r = \frac{1}{2}$, and write $2n$ for n, we shall have

$$\int_0^\infty \frac{X}{\sqrt{x}}\, dx = \frac{\Gamma(2n+1)\,\Gamma(\frac{1}{2})}{\Gamma(2n+\frac{1}{2})} \int_0^\infty \frac{\sin x}{\sqrt{x}}\, dx = \frac{\Gamma(2n+1)}{\Gamma(2n+\frac{1}{2})} \frac{\pi}{\sqrt{2}}.$$

If we put $f(x) = \cos x$, $r = \frac{1}{2}$, and write $2n$ for n,

$$\int_0^\infty \frac{X'}{\sqrt{x}}\, dx = \frac{\Gamma(2n+1)\,\Gamma(\frac{1}{2})}{\Gamma(2n+\frac{1}{2})} \int_0^\infty \frac{\cos x}{\sqrt{x}}\, dx = \frac{\Gamma(2n+1)}{\Gamma(2n+\frac{1}{2})} \frac{\pi}{\sqrt{2}}.$$

In these, X, X' are functions of x given by

$$\sin x = x - \frac{x^3}{\lfloor 3} + \ldots\ldots + (-1)^n \frac{x^{2n}}{\lfloor 2n} X,$$

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \ldots\ldots + (-1)^n \frac{x^{2n}}{\lfloor 2n} X'.$$

Of course the integral can be evaluated for many particular cases.

II. The equation of the director-circle of a conic, whose homogeneous equation of the second degree is $u = 0$, is

$$a^2(\delta_1 - \delta_2)(\delta_1 - \delta_3)u^{\frac{1}{2}} + b^2(\delta_2 - \delta_3)(\delta_2 - \delta_1)u^{\frac{1}{2}} + c^2(\delta_3 - \delta_1)(\delta_3 - \delta_2)u^{\frac{1}{2}} = 0,$$

where a, b, c are the sides of the triangle of reference, and δ_1, δ_2, δ_3 denote the operations $\frac{\delta}{\delta x}$, $\frac{\delta}{\delta y}$, $\frac{\delta}{\delta z}$; the coordinates being areal. If the coordinates be trilinear, the equation will be

$$\frac{\delta^2 u^{\frac{1}{2}}}{\delta x^2} + \frac{\delta^2 u^{\frac{1}{2}}}{\delta y^2} + \frac{\delta^2 u^{\frac{1}{2}}}{\delta z^2} - 2\cos A \frac{\delta^2 u^{\frac{1}{2}}}{\delta y\, \delta z} - 2\cos B \frac{\delta^2 u^{\frac{1}{2}}}{\delta z\, \delta x} - 2\cos C \frac{\delta^2 u^{\frac{1}{2}}}{\delta x\, \delta y} = 0.$$

Taking $u = (l, m, n, l', m', n' \mathbin{\text{\textbigcircle}} x, y, z)^2$, the equation of the two

tangents drawn from $(X : Y : Z)$ is

$$4uU = \left(x\frac{\delta U}{\delta X}+y\frac{\delta U}{\delta Y}+z\frac{\delta U}{\delta Z}\right)^2,$$

and these will be at right angles if

$$4\,(la^2+mb^2+nc^2-2l'\,bc\cos A-2m'\,ca\cos B-2n'\,ab\cos C)\,U$$
$$= a^2\left(\frac{\delta U}{\delta X}\right)^2+\ldots-2bc\cos A\,\frac{\delta U}{\delta Y}\frac{\delta U}{\delta Z}-\ldots-\ldots$$

Writing $b^2+c^2-a^2$ for $2bc\cos A$, &c., and noting that

$$\frac{\delta^2 U}{\delta X^2}=2l,\quad \frac{\delta^2 U}{\delta Y\,\delta Z}=2l',\ \text{&c.},$$

we shall have, for the coefficient of a^2,

$$2U\left(\frac{\delta^2 U}{\delta X^2}-\frac{\delta^2 U}{\delta X\,\delta Y}-\frac{\delta^2 U}{\delta X\,\delta Z}+\frac{\delta^2 U}{\delta X\,\delta Y}\right)$$
$$-\left(\frac{\delta U}{\delta X}\right)^2+\frac{\delta U}{\delta X}\frac{\delta U}{\delta Y}+\frac{\delta U}{\delta X}\frac{\delta U}{\delta Z}-\frac{\delta U}{\delta Y}\frac{\delta U}{\delta Z};$$

but, if $U=v^2$, and we now write x, y, z in place of X, Y, Z,

$\frac{\delta^2 U}{\delta X^2}$ becomes $2v\frac{\delta^2 v}{\delta x^2}+2\left(\frac{\delta v}{\delta x}\right)^2$, and $2U\frac{\delta^2 U}{\delta X^2}-\left(\frac{\delta U}{\delta X}\right)^2$ becomes $4v^3\frac{\delta^2 v}{\delta x^2}$,

and $2U\frac{\delta^2 U}{\delta X\,\delta Y}-\frac{\delta U}{\delta X}\frac{\delta U}{\delta Y}$ becomes $4v^3\frac{\delta^2 v}{\delta x\,\delta y}$,

and similarly for the other terms, or the coefficient of a^2 is

$$4v^3\left(\frac{\delta^2 v}{\delta x^2}-\frac{\delta^2 v}{\delta x\,\delta y}-\frac{\delta^2 v}{\delta x\,\delta z}+\frac{\delta^2 v}{\delta y\,\delta z}\right);$$

and the equation may be written

$$a^2(\delta_1-\delta_2)(\delta_1-\delta_3)u^{\frac12}+b^2(\delta_2-\delta_3)(\delta_2-\delta_1)u^{\frac12}+c^2(\delta_3-\delta_1)(\delta_3-\delta_2)u^{\frac12}=0.$$

With trilinear coordinates, the condition for rectangular tangents is

$$4\,(l+m+n-2l'\cos A-2m'\cos B-2n'\cos C)\,U$$
$$=\left(\frac{\delta U}{\delta X}\right)^2+\ldots+\ldots-2\,\frac{\delta U}{\delta Y}\frac{\delta U}{\delta Z}\cos A-\ldots,$$

$$\text{or } 2U\frac{\delta^2 U}{\delta X^2}-\left(\frac{\delta U}{\delta X}\right)^2+\ldots+\ldots=2\cos A\left(2U\frac{\delta^2 U}{\delta Y\,\delta Z}-\frac{\delta U}{\delta Y}\frac{\delta U}{\delta Z}\right)+\ldots+\ldots,$$

which is, by the same transformation as before, reduced to

$$\frac{\delta^2 u^{\frac12}}{\delta x^2}+\frac{\delta^2 u^{\frac12}}{\delta y^2}+\frac{\delta^2 u^{\frac12}}{\delta z^2}-2\frac{\delta^2 u^{\frac12}}{\delta y\,\delta z}\cos A-2\frac{\delta^2 u^{\frac12}}{\delta z\,\delta x}\cos B-2\frac{\delta^2 u^{\frac12}}{\delta x\,\delta y}\cos C=0.$$

In Cartesian coordinates, ω being the angle between the axes, the director circle is $\quad\frac{\delta^2 u^{\frac12}}{\delta x^2}+\frac{\delta^2 u^{\frac12}}{\delta y^2}=2\frac{\delta^2 u^{\frac12}}{\delta x\,\delta y}\cos\omega.$

NOTE.—*The equations of the foci of a conic whose rational equation of the second degree in areal coordinates is* $u = 0$.

I have, on a former occasion, communicated to the Society a compact form for the equations for the foci referred to Cartesian coordinates inclined at an angle ω, viz.,

$$\frac{\delta^2 u^{\frac{1}{2}}}{\delta x^2} = \frac{\delta^2 u^{\frac{1}{2}}}{\delta y^2} = \frac{\delta^2 u^{\frac{1}{2}}}{\delta x\, \delta y} \sec \omega$$

The corresponding equations for areal coordinates are

$$\frac{\frac{\delta^2 u^{\frac{1}{2}}}{\delta y^2} + \frac{\delta^2 u^{\frac{1}{2}}}{\delta z^2} - 2\frac{\delta^2 u^{\frac{1}{2}}}{\delta y\, \delta z}}{a^2} = \frac{\frac{\delta^2 u^{\frac{1}{2}}}{\delta z^2} + \frac{\delta^2 u^{\frac{1}{2}}}{\delta x^2} - 2\frac{\delta^2 u^{\frac{1}{2}}}{\delta x\, \delta x}}{b^2} = \frac{\frac{\delta^2 u^{\frac{1}{2}}}{\delta x^2} + \frac{\delta^2 u^{\frac{1}{2}}}{\delta y^2} - 2\frac{\delta^2 u^{\frac{1}{2}}}{\delta x\, \delta y}}{c^2},$$

where a, b, c are the sides of the triangle of reference. These forms are most readily proved by taking the axes of Cartesian coordinates x, y along the sides CA, CB of the triangle of reference ABC. Let (X, Y, Z) be the areal coordinates, and (x, y) the Cartesian coordinates of the same point P; then

$$X = \frac{\Delta BPC}{\Delta ABC} = \frac{x}{b}; \text{ so } Y = \frac{y}{a}, \text{ and } Z = 1 - \frac{x}{b} - \frac{y}{a}.$$

Hence
$$\frac{\delta v}{\delta x} = \frac{\delta v}{\delta X}\frac{\delta X}{\delta x} + \frac{\delta v}{\delta Z}\frac{\delta Z}{\delta x} = \frac{1}{b}\left(\frac{\delta v}{\delta X} - \frac{\delta v}{\delta Z}\right),$$

and, similarly,
$$\frac{\delta v}{\delta y} = \frac{1}{a}\left(\frac{\delta v}{\delta Y} - \frac{\delta v}{\delta Z}\right);$$

and the equations for the foci in areal coordinates will therefore be (remembering that $\omega = C$)

$$\frac{1}{a^2}\left(\frac{\delta}{\delta Y} - \frac{\delta}{\delta Z}\right)^2 U^{\frac{1}{2}} = \frac{1}{b^2}\left(\frac{\delta}{\delta X} - \frac{\delta}{\delta Z}\right)^2 U^{\frac{1}{2}}$$
$$= \frac{1}{ab\cos C}\left(\frac{\delta}{\delta X} - \frac{\delta}{\delta Z}\right)\left(\frac{\delta}{\delta Y} - \frac{\delta}{\delta Z}\right) U^{\frac{1}{2}},$$

each of which is therefore equal to

$$\frac{\left\{\left(\frac{\delta}{\delta Y} - \frac{\delta}{\delta Z}\right)^2 + \left(\frac{\delta}{\delta X} - \frac{\delta}{\delta Z}\right)^2 - 2\left(\frac{\delta}{\delta X} - \frac{\delta}{\delta Z}\right)\left(\frac{\delta}{\delta Y} - \frac{\delta}{\delta Z}\right)\right\} U^{\frac{1}{2}}}{a^2 + b^2 - 2ab\cos C},$$

or to
$$\frac{\frac{\delta^2 U^{\frac{1}{2}}}{\delta X^2} + \frac{\delta^2 U^{\frac{1}{2}}}{\delta Y^2} - 2\frac{\delta^2 U^{\frac{1}{2}}}{\delta X\, \delta Y}}{c^2}.$$

The equations for trilinear coordinates can be deduced immediately from these, writing aX, bY, cZ for X, Y, Z respectively (they are proportional only, but the equations are homogeneous). They are therefore $\left(c\frac{\delta}{\delta Y} - b\frac{\delta}{\delta Z}\right)^2 U^{\frac{1}{2}} = \left(a\frac{\delta}{\delta Z} - c\frac{\delta}{\delta X}\right)^2 U^{\frac{1}{2}} = \left(b\frac{\delta}{\delta X} - a\frac{\delta}{\delta Y}\right)^2 U^{\frac{1}{2}}$.

On the Vibrations of an Elastic Sphere. By Horace Lamb, M.A.

[*Read May 11th*, 1882.]

The following paper contains an examination into the nature of the fundamental modes of vibration of an elastic sphere by the method employed in a previous communication, "On the Oscillations of a Viscous Spheroid."* The problem here considered is one of considerable theoretic interest, being as yet the only case in which the vibrations of an elastic solid whose dimensions are all finite have been discussed with any attempt at completeness. I have therefore thought it worth while to go into considerable detail in the interpretation of the results, and have endeavoured, by numerical calculations and the construction of diagrams, to present these in as definite and intelligible a form as possible. I find that some of my results (the most important being the general classification of the fundamental modes given in § 5 below) have been already obtained by Paul Jaerisch, of Breslau, in *Crelle*, t. lxxxviii. (1879); but the methods employed, as well as the form in which the results in question are expressed, are entirely different in the two investigations.

1. For convenience of reference, I place together at the outset several analytical results which will be required.

It has been shown (*Proc. Lond. Math. Soc.*, Dec. 1881), that the solution of the following system of equations—

$$\left.\begin{gathered}(\nabla^2+k^2)\,u=0,\quad (\nabla^2+k^2)\,v=0,\quad (\nabla^2+k^2)\,w=0\\ \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0\end{gathered}\right\}\quad\ldots\ldots(1),$$

where $\nabla^2 = d^2/dx^2+d^2/dy^2+d^2/dz^2$, and k is a constant, is given by the formulæ

$$\left.\begin{aligned}
u &= \Sigma\left\{\psi_n(kr)\left(y\frac{d\chi_n}{dz}-z\frac{d\chi_n}{dy}\right)+\psi_{n-1}(kr)\frac{d\phi_n}{dx}\right.\\
&\qquad\left.-\frac{n}{n+1}\,\frac{k^2r^{2n+3}}{2n+1\,.\,2n+3}\,\psi_{n+1}(kr)\,\frac{d}{dx}\,\frac{\phi_n}{r^{2n+1}}\right\},\\
v &= \Sigma\left\{\psi_n(kr)\left(z\frac{d\chi_n}{dx}-x\frac{d\chi_n}{dz}\right)+\psi_{n-1}(kr)\frac{d\phi_n}{dy}\right.\\
&\qquad\left.-\frac{n}{n+1}\,\frac{k^2r^{2n+3}}{2n+1\,.\,2n+3}\,\psi_{n+1}(kr)\,\frac{d}{dy}\,\frac{\phi_n}{r^{2n+1}}\right\},\\
w &= \Sigma\left\{\psi_n(kr)\left(x\frac{d\chi_n}{dy}-y\frac{d\chi_n}{dx}\right)+\psi_{n-1}(kr)\frac{d\phi_n}{dz}\right.\\
&\qquad\left.-\frac{n}{n+1}\,\frac{k^2r^{2n+3}}{2n+1\,.\,2n+3}\,\psi_{n+1}(kr)\,\frac{d}{dz}\,\frac{\phi_n}{r^{2n+1}}\right\}
\end{aligned}\right\}\quad\ldots\ldots(2).$$

* "Proceedings of the London Mathematical Society," Vol. xiii., pp. 51–66, Nos. 183, 184.

The symbols χ_n, ϕ_n here stand for solid harmonics of algebraical degree n; $r = (x^2+y^2+z^2)^{\frac{1}{2}}$; and ψ_n is defined by the equation

$$\psi_n(\theta) = 1 - \frac{\theta^2}{2.2n+3} + \frac{\theta^4}{2.4.2n+3.2n+5} - \&c. \quad \ldots\ldots\ldots(3).$$

This function possesses the following properties, which will be of frequent application

$$\psi_n'(\theta) = -\frac{\theta}{2n+3}\psi_{n+1}(\theta) \quad \ldots\ldots\ldots(4),$$

$$\psi_n(\theta) + \frac{\theta}{2n+1}\psi'(\theta) = \psi_{n-1}(\theta) \quad \ldots\ldots\ldots(5),$$

$$\psi_n(\theta) - \psi_{n-1}(\theta) = \frac{\theta^2}{2n+1.2n+3}\psi_{n+1}(\theta) \quad \ldots\ldots\ldots(6).$$

In order that (2) should give the solution of the system (1) with complete generality, it is necessary that the sign Σ of summation should embrace both positive and negative values of n; but in the present application the condition that the motion must be finite at the centre of the sphere (which is taken as origin of coordinates) restricts us to positive values of n. We then have

$$\psi_n(\theta) = (-)^n.1.3.5\ldots 2n+1.\left(\frac{1}{\theta}\frac{d}{d\theta}\right)^n \frac{\sin\theta}{\theta} \quad \ldots\ldots\ldots(7).*$$

Another formula for ψ_n is

$$\psi_n(\theta) = \frac{1.3.5\ldots 2n+1}{\theta^{n+1}}\left\{H_n \sin\left(\theta - \frac{n\pi}{2}\right) + K_n \cos\left(\theta - \frac{n\pi}{2}\right)\right\} \quad \ldots(8),$$

where $$H_n = 1 - \frac{n-1.n.n+1.n+2}{2.4.\theta^2} + \&c.,$$

$$K_n = \frac{n.n+1}{2.\theta} - \frac{n-2.n-1.n.n+1.n+2.n+3}{2.4.6.\theta^3} + \&c.$$

If we multiply the three equations (2) by x, y, z in order, and add, we find

$$xu+yv+zw = \Sigma n\psi_n(kr).\phi_n \quad \ldots\ldots\ldots(9),$$

the reduction being effected by means of (6), and of known properties of spherical harmonics.

Again, if we write

$$2\xi = \frac{dw}{dy} - \frac{dv}{dz}, \quad 2\eta = \frac{du}{dz} - \frac{dw}{dx}, \quad 2\zeta = \frac{dv}{dx} - \frac{du}{dy} \quad \ldots\ldots\ldots(10),$$

we find

$$2\xi = -\Sigma\frac{k^2}{2n+1}\psi_n(kr)\left(y\frac{d\phi_n}{dz} - z\frac{d\phi_n}{dy}\right)$$

$$-\Sigma(n+1)\left\{\psi_{n-1}(kr)\frac{d\chi_n}{dx} - \frac{n}{n+1}\frac{k^2r^{2n+3}}{2n+1.2n+3}\psi_{n+1}(kr)\frac{d}{dx}\frac{\chi_n}{r^{2n+1}}\right\}$$

$$\ldots\ldots\ldots(11),$$

* A formula equivalent to this is given by Heine, *Kugelfunctionen*, t. i., p. 240. The formula (8) has been given in slightly different forms by various writers: it appears to be due substantially to Poisson.

with symmetrical formulæ for 2η, 2ζ. In the reduction, use has been made of (4), (5), and of the following formula of Spherical Harmonics,

$$x\phi_n = \frac{r^2}{2n+1}\left\{\frac{d\phi_n}{dx} - r^{2n+1}\frac{d}{dx}\frac{\phi_n}{r^{2n+1}}\right\} \quad \ldots\ldots\ldots\ldots\ldots(12).$$

The exchange of *rôles* between the harmonics ϕ_n, χ_n, in passing from (2) to (11), is noticeable.

By comparing (2), (10), and (11), we see that (2) may be written in the forms

$$u = \frac{dN}{dy} - \frac{dM}{dz}, \quad v = \frac{dL}{dz} - \frac{dN}{dx}, \quad w = \frac{dM}{dx} - \frac{dL}{dy} \quad \ldots\ldots\ldots(13),$$

where $$L = 2\xi/k^2, \quad M = 2\eta/k^2, \quad N = 2\zeta/k^2 \quad \ldots\ldots\ldots\ldots\ldots(14).$$

2. The equations of motion of a homogeneous isotropic elastic solid free from external force may be written

$$\rho\frac{d^2\alpha}{dt^2} = \mathfrak{m}\frac{d\delta}{dx} + \mathfrak{n}\nabla^2\alpha, \quad \rho\frac{d^2\beta}{dt^2} = \mathfrak{m}\frac{d\delta}{dy} + \mathfrak{n}\nabla^2\beta, \quad \rho\frac{d^2\gamma}{dt^2} = \mathfrak{m}\frac{d\delta}{dz} + \mathfrak{n}\nabla^2\gamma \ldots(15).$$

The notation is that of Thomson and Tait, except that $\mathfrak{m}$, $\mathfrak{n}$ are written for the m, n of these writers; viz., α, β, γ are the component displacements, and δ, $= d\alpha/dx + d\beta/dy + d\gamma/dz$, is the dilatation, at the point (x, y, z) of the solid, ρ is the density, $\mathfrak{n}$ the rigidity, and $\mathfrak{m}$ a constant, such that $\mathfrak{m} - \frac{1}{3}\mathfrak{n}$ is the resilience of volume. If the state of stress at the point (x, y, z) be expressed in the usual way by the symbols p_{xx}, p_{xy}, p_{xz}, &c., we have

$$\left.\begin{aligned} p_{xx} &= (\mathfrak{m}-\mathfrak{n})\,\delta + 2\mathfrak{n}\frac{d\alpha}{dx}, & p_{yz} &= \mathfrak{n}\left(\frac{d\beta}{dz} + \frac{d\gamma}{dy}\right), \\ p_{yy} &= (\mathfrak{m}-\mathfrak{n})\,\delta + 2\mathfrak{n}\frac{d\beta}{dy}, & p_{zx} &= \mathfrak{n}\left(\frac{d\gamma}{dx} + \frac{d\alpha}{dz}\right), \\ p_{zz} &= (\mathfrak{m}-\mathfrak{n})\,\delta + 2\mathfrak{n}\frac{d\gamma}{dz}, & p_{xy} &= \mathfrak{n}\left(\frac{d\alpha}{dy} + \frac{d\beta}{dx}\right), \end{aligned}\right\} \quad \ldots\ldots\ldots(16).$$

If the centre of the sphere be taken as origin of coordinates, the conditions that the surface should be free from stress are

$$xp_{xx} + yp_{xy} + zp_{xz} = 0, \quad xp_{yx} + yp_{yy} + zp_{yz} = 0, \quad xp_{zx} + yp_{zy} + zp_{zz} = 0 \ldots(17),$$

when $r = a$, the radius of the sphere.

If we substitute from (16), these may be put in the form

$$\frac{\mathfrak{m}-\mathfrak{n}}{\mathfrak{n}}x\delta + \left(r\frac{d}{dr} - 1\right)\alpha + \frac{d}{dx}(x\alpha + y\beta + z\gamma) = 0, \text{ \&c. \&c.} \quad \ldots(18),$$

where d/dr denotes a differentiation along the radius vector.

3. To determine the fundamental modes, we assume, as usual, that

α, β, γ all vary as e^{ipt}, where $i^2 = -1$. The equations (15) then become

$$-p^2\rho\alpha = \mathfrak{m}\frac{d\delta}{dx} + \mathfrak{n}\nabla^2\alpha, \quad -p^2\rho\beta = \mathfrak{m}\frac{d\delta}{dy} + \mathfrak{n}\nabla^2\beta,$$

$$-p^2\rho\gamma = \mathfrak{m}\frac{d\delta}{dz} + \mathfrak{n}\nabla^2\gamma \quad \ldots\ldots\ldots\ldots\ldots\ldots(19).$$

Differentiating these with respect to x, y, z in order, and adding, we find

$$(\nabla^2 + h^2)\,\delta = 0 \quad \ldots\ldots\ldots\ldots\ldots\

where P_n, A_n, B_n, C_n, D_n are certain constants whose precise values will be investigated further on. These conditions are to hold over the surface of the sphere $r = a$. I proceed to show that they require that

$$P_n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28),$$

$$A_n\omega_n + C_n\phi_n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (29),$$

and

$$B_n\omega_n + D_n\phi_n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30).$$

In the first place, if we multiply (27) by x, y, z in order, and add, we obtain, equating separately to zero the terms involving surface-harmonics of the same order n,

$$n(A_n\omega_n + C_n\phi_n) - (n+1)(B_n\omega_n + D_n\phi_n) = 0 \ldots\ldots\ldots\ldots (31).$$

Again, consider the function

$$U = \Sigma\left\{ P_n\left(y\frac{d\chi_n}{dz} - z\frac{d\chi_n}{dy}\right) + A_n\frac{d\omega_n}{dx} + C_n\frac{d\phi_n}{dx} + \left(\frac{r}{a}\right)^{2n+3}\left(B_n\frac{d}{dx}\frac{\omega_n}{r^{2n+1}} + D_n\frac{d}{dx}\frac{\omega_n}{r^{2n+1}}\right)\right\}.$$

This is finite and continuous, and satisfies the equation $\nabla^2 U = 0$ throughout the interior of the sphere $r = a$, and vanishes [in virtue of (27)] at the surface. Its first derivatives are also finite and continuous. Hence we must have $U = 0$ throughout the sphere. In like manner, we have $V = 0$, $W = 0$, where V, W are obtained from U by cyclical interchange of the letters x, y, z. Hence, differentiating the three equations $U = 0$, $V = 0$, $W = 0$ with respect to x, y, z in order, and adding, we find $\Sigma(2n+3)(n+1)(B_n\omega_n + D_n\phi_n) = 0$, and thence (30). Substituting in (31), we see that (29) must also hold, and the original equations (27) then show that $P_n = 0$.

5. It appears from the foregoing argument that the fundamental modes of vibration fall naturally into two distinct classes.

The modes of the first class are obtained by making $n = 0, 1, 2, 3$, &c., in the formulæ

$$\left.\begin{aligned} \alpha &= \psi_n(kr)\left(y\frac{d\chi_n}{dz} - z\frac{d\chi_n}{dx}\right), \\ \beta &= \psi_n(kr)\left(z\frac{d\chi_n}{dx} - x\frac{d\chi_n}{dz}\right), \\ \gamma &= \psi_n(kr)\left(x\frac{d\chi_n}{dy} - y\frac{d\chi_n}{dx}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (32).$$

For any given value of n the admissible values of k are determined by the superficial conditions (18), which yield an equation in ka (written above for shortness in the form $P_n = 0$), whose precise form will be ascertained presently.

When the values of k are known, the corresponding frequencies

$(p/2\pi)$ are determined by (23). In any mode of this class, the lines of motion (to borrow a term from Hydrodynamics) lie on a series of spherical surfaces having their common centre at the origin; viz., they are the lines in which these surfaces are intersected by the cones $\chi_n/r^n = \text{const.}$; in other words, the system of lines of motion lying on any one of the spherical surfaces in question are the contour-lines of the harmonic χ_n on that surface. The motions on any two such surfaces are geometrically similar, differing only in amplitude. The amplitude of vibration at any point is proportional to the corresponding value of $\psi_n(kr) \cdot d\chi_n/d\epsilon$, where $d\chi_n$ is the change in the value of χ_n produced by an infinitesimal displacement in a direction at right angles to the line of motion and to the radius vector, the displacement being measured by the angle $d\epsilon$ which it subtends at the centre of the sphere. The nodal surfaces consist of the spheres defined by $\psi_n(kr) = 0$, and of the cones for which $d\chi_n/d\epsilon = 0$. A closer examination of the nature of the motion in one or two of the more interesting cases will be made below.

The vibrations of the second class are of a more complex character. The fundamental modes are obtained by making $n = 0, 1, 2, 3$, &c., in the formulæ

$$\left.\begin{aligned}
\alpha &= -\frac{1}{h^2}\frac{d\delta_n}{dx} + \psi_{n-1}(kr)\frac{d\phi_n}{dx} - \frac{n}{n+1}\frac{k^2r^{2n+3}}{2n+1\,.\,2n+3}\psi_{n+1}(kr)\frac{d}{dx}\frac{\phi_n}{r^{2n+1}} \\
\beta &= -\frac{1}{h^2}\frac{d\delta_n}{dy} + \psi_{n-1}(kr)\frac{d\phi_n}{dy} - \frac{n}{n+1}\frac{k^2r^{2n+3}}{2n+1\,.\,2n+3}\psi_{n+1}(kr)\frac{d}{dy}\frac{\phi_n}{r^{2n+1}} \\
\gamma &= -\frac{1}{h^2}\frac{d\delta_n}{dz} + \psi_{n-1}(kr)\frac{d\phi_n}{dz} - \frac{n}{n+1}\frac{k^2r^{2n+3}}{2n+1\,.\,2n+3}\psi_{n+1}(kr)\frac{d}{dz}\frac{\phi_n}{r^{2n+1}}
\end{aligned}\right\} \quad \ldots(33),$$

where $$\delta_n = \psi_n(hr) \cdot \omega_n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(34).$$

The superficial conditions yield two equations, of which (29) and (30) are the abbreviated forms. The elimination of the ratio ω_n/ϕ_n between these leads to an equation involving both ha and ka; but, since the ratio h/k is determined by the elastic properties of the substance, viz., it $= \sqrt{\{n/(m+n)\}}$, there is virtually but one unknown. The vibration at any point P may be resolved into two components, one in the direction of the radius vector, the other at right angles to it. The value of the radical component is

$$(x\alpha + y\beta + z\gamma)/r = -\frac{1}{h^2}\frac{d\delta_n}{dr} + \frac{n}{r}\psi_n(kr)\,\phi_n \ldots\ldots\ldots(34a).$$

The second or tangential component is at right angles to that cone of the system $\phi_n/r^n = \text{const.}$ which passes through P, and its value is $F(r) \cdot d\phi_n/d\epsilon$, where $F(r)$ is a certain function of r, and $d\epsilon$ has the

same meaning as before.* Except in the species $n = 0$ there are no nodal surfaces (*i.e.*, surfaces at every point of which there is absolute rest), but there may be nodal *lines*.† There are, however, certain surfaces which are of great use in helping us to understand the general character of the vibration in any given case, viz., the spheres and cones at every point of which the normal motion is zero. The spherical surfaces of this kind are obtained by equating the right-hand side of (34*a*) to zero; the cones are those for which $d\phi_n/d\epsilon = 0$.

6. It may be well to point out, at this stage, that the results of our analysis are not in accordance with the views advanced by Lamé‡ as to the nature of the fundamental modes of vibration of elastic solids in general. Expressed in modern terminology, Lamé's reasoning is substantially as follows. From the equations (15), we deduce by differentiation

$$\frac{d^2\xi}{dt^2} = \frac{\mathfrak{n}}{\rho}\nabla^2\xi, \quad \frac{d^2\eta}{dt^2} = \frac{\mathfrak{n}}{\rho}\nabla^2\eta, \quad \frac{d^2\zeta}{dt^2} = \frac{\mathfrak{n}}{\rho}\nabla^2\zeta \ldots\ldots\ldots\ldots(35),$$

and

$$\frac{d^2\delta}{dt^2} = \frac{\mathfrak{m}+\mathfrak{n}}{\rho}\nabla^2\delta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(36),$$

where δ denotes, as before, the dilatation, and ξ, η, ζ are the components of the elementary rotation, at the point (x, y, z); viz.,

$$2\xi = d\gamma/dy - d\beta/dz, \text{ \&c.}$$

From these equations, and from a known theorem of Kinematics, it results,§ that any arbitrary disturbance, originated in any part of an elastic solid, breaks up in general into two distinct waves which travel with different velocities. We have, in the first place, a wave of pure rotation, unaccompanied by dilatation, propagated with a velocity $= \sqrt{(\mathfrak{n}/\rho)}$; and superposed on this there is a wave of pure dilatation, without rotation, travelling with the velocity $\sqrt{\{(\mathfrak{m}+\mathfrak{n})/\rho\}}$. In *plane* waves of these two kinds, the vibrations are respectively parallel and perpendicular to the wave front, and a similar statement holds good approximately in the general case. Hence the two kinds of waves are

* The simplest way of verifying these statements is by a special choice of axes of coordinates, *e.g.*, take x in the direction of the radius vector, and y parallel to the normal at P to the cone $\phi_n/r^n = \text{const.}$ The value of $F(r)$ is

$$F(r) = \frac{1}{r}\left\{-\Upsilon\psi_n(kr) + \frac{(2n+1)\,\psi_{n-1}(kr) - n\psi_n(kr)}{n+1}\right\},$$

where Υ denotes the ratio of ω_n/h^2 to ϕ_n, obtained from (29) or (30).

† The nodal lines are of two classes, viz., (1) those formed by the intersections of the spheres and cones whose description follows above; and (2) those formed by the intersections of the spheres $F(r) = 0$ with the cones $\phi_n = 0$. The distinction between the two classes is illustrated in Fig. 4.

‡ *Théorie Mathématique de l'Élasticité*, 11me leçon.

§ Stokes, *Camb. Trans.*, t. ix., p. 12 (1849).

conveniently characterized as waves of *transverse* and of *longitudinal* vibration respectively. Now any vibration of a finite elastic solid may be conceived as made up of systems of waves travelling to and fro in it, and undergoing continual reflection at the boundary. Lamé proceeds to infer that the fundamental modes will in all cases fall into two distinct classes, in which the constituent waves are of transverse and of longitudinal displacement, respectively. In other words, in the modes of the former class there would be no change of volume, whilst in those of the latter class the motion would be irrotational. Now, although in the classification of § 5, the vibrations of the first class are purely "transversal," those of the second class are, if we except the particular case of the radial vibrations [obtained by making $n=0$ in the formulæ (33), (34)], not irrotational. The error in Lamé's reasoning consists in the tacit assumption that a wave undergoes no change of character on reflection at the boundary of the solid. It was pointed out by Green,* in 1837, that a wave of transversal vibrations, falling on the boundary of an elastic medium, will in general give rise to *two* reflected waves, one of transversal, the other of longitudinal displacement, and the complementary statement is of course also true.

The fact that the radial vibrations are the only purely "longitudinal" modes (in Lamé's sense of the word) of which an elastic sphere is capable, seems to have been first recognized by Jaerisch. The same writer has pointed out the existence of the modes which I have termed "of the second class," and has given at the end of his paper formulæ (in polar coordinates) which can be shown to be equivalent to (33), (34).

Vibrations of the First Class (Transversal Vibrations).

7. The formulæ (32) make $\delta = 0$, and $x\alpha + y\beta + z\gamma = 0$, so that the superficial conditions (18) reduce to $(rd/dr - 1)\,\alpha = 0$, &c., &c., whence

$$ka\psi_n'(ka) + (n-1)\,\psi_n(ka) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (37).\dagger$$

This agrees with a result of Herr Jaerisch. I proceed to examine some of the simpler cases.

Species $n = 1$ (*Rotatory Vibrations*).

If we take the axis of the harmonic χ_1 as axis of x, writing $\chi_1 = x$,‡ the formulæ (31) become

$$\alpha = 0, \quad \beta = \psi_1(kr)\,.\,z, \quad \gamma = -\,\psi_1(kr)\,.\,y.$$

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* *Camb. Trans.*, t. vii., pp. 15, 16; or *Collected Papers*, p. 260.

† Since ψ_n is identical with the infinite series which occurs in the expression for the Bessel's Function of order $n+\frac{1}{2}$, it follows from known properties of these functions that the roots of this equation in ka are all real. See Todhunter, *Functions of Laplace, &c.*, § 400.

‡ The time-factor is omitted here (and elsewhere in what follows) for shortness.

Each of the infinitely thin concentric spherical strata of which the sphere may be supposed made up oscillates as a whole about the axis of the harmonic χ_1 as axis of rotation. The amplitude of the angular displacement of any such stratum is proportional to the corresponding value of $\psi_1(kr)$. The equation (37) reduces to $ka\psi_1'(ka) = 0$, *i.e.*, by (4), to $k^3a^2\psi_2(ka) = 0$. The first root is $ka = 0$; this gives an infinitely long period, and corresponds to the rotation of the sphere as a whole. The remaining roots are to be obtained from $\psi_2(ka) = 0$. By means of (7) or (8), we bring this into the form

$$\tan\theta = \frac{3\theta}{3-\theta^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(38),$$

where $\theta = ka$. The approximate positions of the roots of (38) can be found by a well-known graphical method. When s is at all large, the value of the s^{th} root* only falls a little short of $(s+1)\pi$. The calculation of the roots can be carried to any desired degree of accuracy by the method explained by Fourier,† for the equation (40) below; or we may (in the case of the higher roots) make use of a series, as in Rayleigh's *Sound*, t. ii., p. 233. I find, for the first six roots,

$$ka/\pi = 1{\cdot}8346,\ 2{\cdot}8950,\ 3{\cdot}9225,\ 4{\cdot}9385,\ 5{\cdot}9489,\ 6{\cdot}9563 \ \ldots(39).$$

It may be remarked that the value of ka/π for any fundamental mode is equal to T_0/τ, where τ is the period of oscillation, and T_0 is the time which would be occupied by a wave of *transversal* displacement in travelling a distance equal to the diameter of the sphere. In the mode corresponding to any one of the roots (39) after the first, the roots of lower order give the positions of the internal spherical *loop-surfaces*, *i.e.*, surfaces across which there is no stress. Thus in the second mode there is a spherical loop whose radius r is given by

$$r/a = (1{\cdot}8346)/(2{\cdot}8950),$$

or $r = {\cdot}6337a$. The positions of the spherical *nodes* are given by

$$\psi_1(kr) = 0, \text{ or } \tan\theta = \theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(40),$$

where $\theta = kr$. The solutions of this are known‡ to be

$$kr/\pi = 1{\cdot}4303,\ 2{\cdot}4590,\ 3{\cdot}4709,\ 4{\cdot}4774,\ 5{\cdot}4818,\ 6{\cdot}4844 \ \ldots(41).$$

Species $n = 2$.

The equation (37) becomes in this case

$$ka\,\psi_2'(ka) + \psi_2(ka) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(42).$$

* We do not count the zero root, which is irrelevant, having been introduced by the algebraical manipulation. The same remark applies to other equations of a similar character which occur further on.

† *Theory of Heat* (Freeman's translation), p. 271.

‡ Schwerd, quoted by Verdet, *Optique Physique*, t. i., p. 266.

Substituting the value of ψ_2 from (7) or (8), and reducing, we obtain

$$\cot\theta = \frac{5\theta^2-12}{\theta^3-12\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (43),$$

where $\theta = ka$. We readily find that the first root of this lies between $\pi/2$ and π, and that, when s is large, the s^{th} root falls a little short of $(s+\frac{1}{2})\,\pi$. The calculation of the first root is a little troublesome, and is best effected by one of the methods explained by Lord Rayleigh in the Society's *Proceedings*, t. v., p. 119. I find, for the first six roots,

$$ka/\pi = \cdot 7961,\ 2\cdot 2715,\ 3\cdot 3469,\ 4\cdot 3837,\ 5\cdot 4059,\ 6\cdot 4209 \ \ldots (44).$$

As before, in the modes corresponding to any one of these roots after the first, the inferior roots give the positions of the internal spherical loops. The positions of the spherical nodes are given by $\psi_2(kr) = 0$, the roots of which have been already stated in (39).

The character of the vibration depends on the form of χ_2. The fundamental types of spherical harmonic of the second degree are the zonal harmonic, and the sectorial harmonic with two nodal diametral planes. Taking first the case of the zonal harmonic, we write

$$\chi_2 = 2x^2 - y^2 - z^2;$$

the formulæ (32) then become

$$\alpha = 0, \quad \beta = 6\psi_2(kr)\,.\,xz, \quad \gamma = -\,6\psi_2(kr)\,.\,xy.$$

Every particle oscillates in a direction at right angles to the meridional plane in which it is situate, the amplitude of the angular displacement about the axis of the harmonic being proportional to the corresponding value of $\psi_2(kr)\,.\,x$. The sign of the displacement is reversed as we cross the equatorial plane, which is nodal. In the particular case where there is no spherical node, the motion in its main features somewhat resembles the gravest torsional vibration of a solid cylinder having about equal lateral and longitudinal dimensions. The comparative slowness of the vibration in this case [its frequency is proportional to the slowest of the roots (44)] is thus easily understood.*

The modes of vibration corresponding to the sectorial harmonic may be examined by making $\chi_2 = yz$. The motion is symmetrical with respect to each of the planes $y = 0$, $z = 0$; and the lines of motion corresponding to any surface having its centre at the origin form four systems of oval curves lying in the four lunes into which the surface is divided by the two planes in question.†

* For a cylinder of length $2a$ the corresponding value of ka is ·5. From the results given further on it appears probable that the mode in question is the gravest of all the fundamental modes of the sphere.

† This statement may be generalized so as to apply to all forms of the harmonic χ_2, provided we replace the two planes referred to by two planes drawn through the centre of the sphere at right angles to the two axes of χ_2. It is interesting to trace in this way the transition to the case of the zonal harmonic, when the two axes coincide.

I have not thought it necessary to carry the discussion of the transversal vibrations any further. With the help of the properties stated in § 5, a clear idea of the nature of the motion in some varieties of the third and fourth species may be gathered from Plates VI. to IX., at the end of the first volume of Maxwell's *Electricity*.

Vibrations of the Second Class.

8. These are defined by the equations (33), which we write for shortness in the form

$$\alpha = -\frac{1}{h^2}\frac{d\delta_n}{dx} + \alpha', \quad \beta = -\frac{1}{h^2}\frac{d\delta_n}{dy} + \beta', \quad \gamma = -\frac{1}{h^2}\frac{d\delta_n}{dz} + \gamma' \ldots (45).$$

Since $k^2/h^2 = (\mathfrak{m}+\mathfrak{n})/\mathfrak{n}$, the first of the surface conditions (18) may be written

$$\left(\frac{k^2}{h^2} - 2\right) x\delta_n + \left(r\frac{d}{dr} - 1\right)\alpha + \frac{d}{dx}(x\alpha + y\beta + z\gamma) = 0 \quad \ldots\ldots (46).$$

Let us first substitute in this the parts of α, β, γ which involve δ_n.

We obtain $$\left(\frac{k^2}{h^2} - 2\right) x\delta_n - \frac{2}{h^2} r\frac{d}{dr}\cdot\frac{d\delta_n}{dx} \quad \ldots\ldots (47).$$

Now, $$x\delta_n = \psi_n(hr)\cdot\frac{r^2}{2n+1}\left(\frac{d\omega_n}{dx} - r^{2n+1}\frac{d}{dx}\frac{\omega_n}{r^{2n+1}}\right) \ldots\ldots (48),$$

and $$\frac{d\delta_n}{dx} = \psi_{n-1}(hr)\frac{d\omega_n}{dx} + \frac{h^2r^{2n+3}}{2n+1\,.\,2n+3}\psi_{n+1}(hr)\frac{d}{dx}\frac{\omega_n}{r^{2n+1}} \ldots\ldots (49),$$

the reductions being effected by means of (12) and of the relations (4), (5). Hence

$$r\frac{d}{dr}\frac{d\delta_n}{dx} = \{hr\psi'_{n-1}(hr) + (n-1)\psi_{n-1}(hr)\}\frac{d\omega_n}{dx} + \frac{h^2r^{2n+3}}{2n+1\,.\,2n+3}\{hr\psi'_{n+1}(hr) + (n+1)\psi_{n+1}(hr)\}\frac{d}{dx}\frac{\omega_n}{r^{2n+1}} \ldots (50).$$

Substituting from (48) and (50) in (47), and making $r = a$, we obtain, after a little reduction,

$$A_n\frac{d\omega_n}{dx} + B_n\frac{d}{dx}\frac{\omega_n}{r^{2n+1}} \ldots\ldots (51),$$

where $$A_n = \frac{1}{h^2}\left\{\frac{k^2a^2}{2n+1}\psi_n(ha) - 2(n-1)\psi_{n-1}(ha)\right\} \quad \ldots\ldots (52),$$

$$B_n = -\frac{1}{h^2}\cdot\frac{k^2a^{2n+3}}{2n+1}\left\{\psi_n(ha) + \frac{2(n+2)}{k^2a^2}ha\psi'_n(ha)\right\} \ldots\ldots (53).$$

It remains to calculate the part of (46) depending on α', β', γ'. We have, by (9),

$$x\alpha'+y\beta'+z\gamma' = n\psi_n(kr)\cdot\phi_n,$$

and thence, as in (49),

$$\frac{d}{dx}(x\alpha'+y\beta'+z\gamma') = n\psi_{n-1}(kr)\frac{d\phi_n}{dx}+n\frac{k^2r^{2n+3}}{2n+1\,.\,2n+3}\psi_{n+1}(kr)\frac{d}{dx}\frac{\phi_n}{r^{2n+1}}.$$

Again,

$$\left(r\frac{d}{dr}-1\right)\alpha' = \{kr\psi'_{n-1}(kr)+(n-2)\,\psi_{n-1}(kr)\}\frac{d\phi_n}{dx}$$
$$-\frac{n}{n+1}\frac{k^2r^{2n+3}}{2n+1\,.\,2n+3}\{kr\psi'_{n+1}(kr)+n\psi_{n-1}(kr)\}\frac{d}{dx}\frac{\phi_n}{r^{2n+1}}.$$

Hence the remaining part of the surface condition (46) is found to be

$$C_n\frac{d\phi_n}{dx}+D_n\frac{d}{dx}\frac{\phi_n}{r^{2n+1}} \qquad \text{.....(54),}$$

where
$$C_n = ka\psi'_{n+1}(ka)+2\,(n-1)\,\psi_{n-1}(ka)$$
$$= -\left\{\frac{k^2a^2}{2n+1}\,\psi_n(ka)-2\,(n-1)\,\psi_{n-1}(ka)\right\} \qquad \text{.....(55),}$$

and
$$D_n = -\frac{n}{n+1}\frac{k^2a^{2n+3}}{2n+1\,.\,2n+3}\{ka\psi'_{n+1}(ka)-\psi_{n+1}(ka)\}$$
$$= -\frac{n}{n+1}\frac{k^2a^{2n+3}}{2n+1}\left\{\psi_n(ka)+\frac{2\,(n+2)}{k^2a^2}\,ka\psi'_n(ka)\right\} \qquad \text{.....(56),}$$

the transformations being effected by means of the relations (4), (5).

It follows from the reasoning of § 4, that the final equation to determine the frequencies of vibration is to be found* by eliminating the ratio ω_n/ϕ_n between the equations

$$A_n\omega_n+C_n\phi_n = 0 \qquad \text{.....(29),}$$
$$B_n\omega_n+D_n\phi_n = 0 \qquad \text{.....(30),}$$

viz., it is
$$B_n/A_n = D_n/C_n \qquad \text{.....(57),}$$

where the values of A_n, B_n, C_n, D_n are to be substituted from (52), (53), (55), (56).

I proceed to examine some of the more important cases.

Species $n=0$ *(Radial Vibrations).*

9. In this case we have α', β', $\gamma'=0$, so that

$$\alpha = -\frac{1}{h^2}\frac{d\delta_0}{dx},\quad \beta = -\frac{1}{h^2}\frac{d\delta_0}{dy},\quad \gamma = -\frac{1}{h^2}\frac{d\delta_0}{dz}$$

* It is here assumed that $n>0$. When $n=0$ we have α', β', $\gamma'=0$, and the process becomes much simpler. See § 9.

where δ_0 is proportional to $\psi_0(hr)$. The motion is therefore symmetrical about the centre of the sphere, being everywhere in the direction of the radius vector. The expression (51) reduces to

$$B_0 \frac{d}{dx} \frac{\omega_0}{r},$$

where ω_0 is a constant, so that the surface conditions yield simply $B_0 = 0$. This may be written

$$\psi_0(\theta) + \frac{4}{\lambda^2 \theta} \psi_0'(\theta) = 0 \quad \text{.....(58),*}$$

where $\theta = ha$, and $\lambda^2, = h^2/k^2$, is a constant depending on the elastic properties of the substance of the sphere, viz., it $= \mathfrak{n}/(\mathfrak{m}+\mathfrak{n})$. If σ denote the ratio of lateral contraction to longitudinal extension when a bar of the substance is stretched lengthwise, we have $\sigma = (\mathfrak{m}-\mathfrak{n})/2\mathfrak{m}$,

and thence $$\lambda^2 = \frac{\frac{1}{2}-\sigma}{1-\sigma}.$$

For actual solids the values of σ may, for all we know, range from 0 to $\frac{1}{2}$,† and therefore λ^2 may range between $\frac{1}{2}$ and 0.

There is no difficulty in showing that, for any admissible value of λ^2, the roots of (58) are all real, and are separated by the roots of $\psi_0(\theta) = 0$, which are $\theta = s\pi$. Writing for $\psi_0(\theta)$ its value $\sin\theta/\theta$, we bring (58) into the form

$$\tan\theta = \frac{4\theta}{4 - \frac{\theta^2}{\lambda^2}} \quad \text{.....(59).}$$

For any assigned value of λ^2 the roots of this can be calculated by methods already indicated. I have computed the values of ha/π corresponding to the first six roots in each of the following cases:—

$$\sigma = 0,\ \tfrac{1}{4},\ \tfrac{3}{10},\ \tfrac{1}{3}.\ ‡$$

The corresponding values of λ^2 are

$$\lambda^2 = \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{2}{7},\ \tfrac{1}{4}.$$

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* The theory of the radial vibrations has been treated by Clebsch. Allowing for difference of notation, the equation (58) will be found to agree with that numbered (48) at p. 59 of his *Theorie der Elasticität*.

† The case $\sigma = 0$ seems to be realized approximately in cork. The value $\sigma = \frac{1}{2}$ corresponds to a substance whose resilience of volume is infinite in comparison with its rigidity.

‡ The values $\sigma = \frac{1}{4}$, $\sigma = \frac{1}{3}$ are those assumed by Poisson and Wertheim respectively as applicable to all truly isotropic substances. The value $\sigma = \frac{3}{10}$ lies between the values (·294 and ·310) found experimentally by Kirchhoff and Everett, respectively, for *steel*. See Everett, *Units and Physical Constants*, p. 54.

The results of the calculation are given in the following table :—

s	$\sigma = 0$	$\sigma = \frac{1}{4}$	$\sigma = \frac{3}{10}$	$\sigma = \frac{1}{3}$
1	·6626	·8160	·8500	·8733
2	1·8909	1·9285	1·9391	1·9470
3	2·9303	2·9539	2·9606	2·9656
4	3·9485	3·9658	3·9707	3·9744
5	4·9590	4·9728	4·9767	4·9796
6	5·9660	5·9774	5·9806	5·9830

The physical meaning of the constant ha/π is similar to that of ka/π, viz., we have $ha/\pi = T_1/\tau$, where T_1 is the time occupied by a wave of *longitudinal* vibration in traversing a space equal to the diameter of the sphere, and τ denotes as before the time of oscillation.

To compare the frequencies of vibration in the above cases with those of the various transversal modes, we should have to calculate the corresponding values of ka/π. These would be obtained by multiplying the numbers in the several columns of the above table by the factors

1·4142, 1·7321, 1·8708, 2,

respectively. It appears that, for any particular mode, the value of ka increases as λ^2 diminishes. This is in accordance with a general principle formulated by Lord Rayleigh.* If, keeping n constant, we increase m, the potential energy corresponding to any deformation which involves a change of density will be increased. It follows, by the principle referred to, that all the modes of the second class will rise in frequency.

In the mode corresponding to any root of (59) after the first, the inferior roots give the positions of the spherical loops. The positions of the spherical nodes are determined by $d\delta_0/dr = 0$, or $\tan hr = hr$. The corresponding values of hr/π are the numbers quoted in (41).

10. In dealing with the higher species, it is convenient to discuss separately, in the first instance, the case where the substance of the sphere is *incompressible* (*i.e.*, $m = \infty$, $\lambda^2 = 0$, $\sigma = \frac{1}{2}$). This is of course an extreme case, but it is very amenable to mathematical treatment, and the results will help us to understand the nature of the various modes in other cases where the actual calculation would be much more difficult. Moreover, as we have just seen, the values of ka obtained on the supposition of incompressibility will be superior limits to the values of ka for the corresponding modes in the general case.

Making, then, $h = 0$ in the formulæ for A_n, B_n, C_n, D_n, we find that

* *Theory of Sound*, t. i., p. 85.

the equations (29), (30) may be written

$$\left\{\frac{\theta^2}{2n+1}-2(n-1)\right\}\frac{\omega_n}{h^2}-\left\{\frac{\theta^2}{2n+1}\psi_n-2(n-1)\psi_{n-1}\right\}\phi_n=0\ldots(60),$$

$$\frac{\omega_n}{h^2}+\frac{n}{n+1}\left\{\psi+\frac{2(n+2)}{\theta}\psi_n'\right\}\phi_n=0\ldots\ldots\ldots\ldots\ldots\ldots(61),$$

where $\theta=ka$, and ψ_n is written in place of $\psi_n(\theta)$ for shortness. It is to be remarked that, though h is zero, ω_n/h^2 is finite; viz., if Π denote the average normal *tension* about the point (x, y, z), we have $\omega_n/h^2=\Pi/p^2\rho$. Eliminating the ratio ω_n/ϕ_n between (60), (61), we obtain an equation to determine θ. This equation can, by means of the relations (4), (5), be put into a great variety of forms, of which the following is perhaps the simplest :—

$$(2n-1)\psi_{n-2}-(2n+1)\psi_{n-1}+2n\psi_n-\frac{4n\,.\,n-1\,.\,n+2}{2n+1\,.\,2n+3}\psi_{n+1}=0\ldots(62).$$

When expanded in rising powers of θ, this equation becomes, after considerable reduction,

$$\Sigma\frac{8m^3+m^2(16n+8)+m(16n^2+4n-10)+4n^3+4n^2-2n-6}{2\,.\,4\ldots2m\,.\,2n+1\,.\,2n+3\ldots2n+2m+3}(-)^m\,.\,\theta^{2m}$$
$$=0\ldots\ldots\ldots\ldots\ldots\ldots(63),$$

the first term being

$$\frac{(n-1)(4n^2+8n+6)}{2n+1\,.\,2n+3}.$$

In the actual calculation of the roots, I have, however, preferred to work out the formulæ for each value of n separately, using the equations (62), (63) only by way of verification.

For any assigned value of n, the character of the vibration will depend of course on the order of the root and on the form of the harmonic ϕ_n.* In the important case of the *zonal* harmonic, we may borrow the method of the Stream-function from Hydrodynamics. If the motion of an incompressible substance take place in a series of planes through the axis of x, and be the same in each such plane, and if u, v, w denote the components of the displacement at the point (x, y, z), we may write†

$$u=\frac{1}{\varpi}\frac{d\Psi}{d\varpi},\quad \mathfrak{v}=-\frac{1}{\varpi}\frac{d\Psi}{dx}\ldots\ldots\ldots\ldots\ldots\ldots(64),$$

where $\varpi=\sqrt{(y^2+z^2)}$, and $\mathfrak{v}$, $=(yv+zw)/\varpi$, is the displacement in the direction of ϖ. The function Ψ represents 2π times the total volume displaced across any surface bounded by the circle whose coordinates

* Or ω_n, since these two harmonics are proportional.

† *Motion of Fluids*, § 103.

are (x, ϖ). Let us apply this to the case where

$$u = \psi_{n-1}(kr)\frac{d\phi_n}{dx} - \frac{n}{n+1}\frac{k^2r^{2n+3}}{2n+1\,.\,2n+3}\psi_{n+1}(kr)\frac{d}{dx}\frac{\phi_n}{r^{2n+1}}, \text{ \&c., \&c.,}$$

ϕ_n being a zonal harmonic with the x-line as axis. The corresponding value (Ψ_1 say) of Ψ is*

$$\Psi_1 = yN - zM \quad \ldots\ldots(65),$$

where M, N are to be found from (14) and (11), by putting $\chi_n = 0$, and omitting the sign Σ; viz.,

$$M = -\frac{1}{n+1}\psi_n(kr)\left(z\frac{d\phi_n}{dx} - x\frac{d\phi_n}{dz}\right),$$

$$N = -\frac{1}{n+1}\psi_n(kr)\left(x\frac{d\phi_n}{dy} - y\frac{d\phi_n}{dx}\right).$$

Substituting in (65), we find

$$\Psi_1 = -\frac{1}{n+1}\psi_n(kr)\left(nx\phi_n - r^2\frac{d\phi_n}{dx}\right) \quad \ldots\ldots(66).$$

The stream-function (Ψ_2 say) corresponding to the case

$$u = -\frac{1}{h^2}\frac{d\omega_n}{dx}, \quad v = -\frac{1}{h^2}\frac{d\omega_n}{dy}, \quad w = -\frac{1}{h^2}\frac{d\omega_n}{dz},$$

where ω_n also is a zonal harmonic with the x-line as axis, is found from this by making $k = 0$ and writing $-\omega_n/h^2$ for ϕ_n, viz., it is

$$\Psi_2 = \frac{1}{h^2}\frac{1}{n+1}\left(nx\omega_n - r^2\frac{d\omega_n}{dx}\right) \quad \ldots\ldots(67).$$

Hence for the fundamental modes of species n we have, when the harmonic ϕ_n is zonal,

$$\Psi = \Psi_1 + \Psi_2 = -\frac{1}{n+1}\left(nx\phi_n - r^2\frac{d\phi_n}{dx}\right)\{\psi_n(kr) - \Upsilon\} \quad \ldots\ldots(68),$$

where Υ denotes the value of the ratio ω_n/h^2 to ϕ_n, to be obtained from (60) or (61). This result may also be expressed in the form

$$\Psi = -\frac{1}{n+1}\varpi\frac{d\phi_n}{d\epsilon}\{\psi_n(kr) - \Upsilon\} \quad \ldots\ldots(68a),$$

where ϵ is the co-latitude. By assigning to Ψ a series of constant values, we get the equations of the lines of motion.

From (34) and (34a) we obtain, since h is now $= 0$,

$$x\alpha + y\beta + z\gamma = n\{\psi_n(kr) - \Upsilon\}\phi_n \quad \ldots\ldots(69).$$

This formula holds independently of the form of ϕ_n. We see that, over

* *Ibid.*, §§ 141, 142.

the spherical surfaces for which $\psi_n(kr) = \Upsilon$, the motion is wholly tangential.

We can now proceed to the examination of particular cases.

Species $n = 1$. $[\sigma = \frac{1}{2}.]$

11. The equations (60), (61) become

$$\frac{\omega_1}{h^2} - \psi_1(\theta) \cdot \phi_1 = 0 \quad \text{.....(70),*}$$

$$\frac{\omega_1}{h^2} + \tfrac{1}{2}\left\{\psi_1(\theta) + \frac{6}{\theta}\psi_1'(\theta)\right\}\phi_1 = 0 \quad \text{.....(71).}$$

Eliminating the ratio ω_1/ϕ_1, we find

$$1 + \frac{2\psi_1'(\theta)}{\theta\psi_1(\theta)} = 0 \quad \text{.....(72).}$$

It is easy to prove that the roots of this are all real, and are separated by the roots of $\psi_1(\theta) = 0$, which have been given in (41). If we substitute the value of $\psi_1(\theta)$ from (7) or (8), (72) becomes

$$\cot\theta = \frac{3\theta^2 - 6}{\theta^3 - 6\theta} \quad \text{.....(73).}$$

I find, for the first six roots of this equation,

$$ka/\pi = 1{\cdot}2319,\ 2{\cdot}3692,\ 3{\cdot}4101,\ 4{\cdot}4310,\ 5{\cdot}4439,\ 6{\cdot}4528 \quad \text{...(74).}$$

The motion is evidently symmetrical about the axis of the harmonic ϕ_1 which we may conveniently take as axis of x, writing $\phi_1 = x$. From (70) we have $\Upsilon = \psi_1(ka)$, so that (68) becomes

$$\Psi = \frac{\varpi^2}{2}\{\psi_1(kr) - \psi_1(ka)\} \quad \text{.....(75).}$$

It appears that at the surface of the sphere the motion is wholly tangential. I have traced, in Figs. 1 and 2, the lines of motion corresponding to equidistant values of Ψ, for the fundamental modes corresponding to the first and second roots of (73). In the mode corresponding to the s^{th} root, there are $s-1$ internal spherical surfaces of zero normal motion whose radii are given by

$$\psi_1(kr) = \psi_1(ka) \quad \text{.....(76);†}$$

there are no spherical nodes (in the strict sense of the term), but there are s nodal *circles* in the equatorial plane. The radii of these circles

* This equation may be otherwise obtained by expressing that the total momentum of the sphere is zero.

† For $s = 2$, I find $kr/\pi = 1{\cdot}465$, whence $r/a = {\cdot}618$.

are defined by $d\Psi/d\varpi = 0$, $x = 0$; *i.e.*, by

$$\psi_1(k\varpi) + \frac{k\varpi}{2}\psi_1'(k\varpi) = \psi_1(ka) \quad \text{.....(77).*}$$

There are no internal spherical loops.

The amplitude of vibration at the centre of the sphere is (on the scale of our formulæ)

$$a_0 = 1 - \psi_1(ka),$$

and the amplitude at any point on the equator is (on the same scale)

$$a_E = \tfrac{1}{2}ka\psi_1'(ka).$$

For the first mode I find $a_E/a_0 = -\cdot 4865$.

Species $n = 2$. $[\sigma = \frac{1}{2}.]$

Writing $n = 2$ in the formulæ (60), (61), we have

$$\left\{\frac{\theta^2}{5} - 2\right\}\frac{\omega_2}{k^2} - \left\{\frac{\theta^2}{5}\psi_2(\theta) - 2\psi_1(\theta)\right\}\phi_2 = 0 \quad \text{.....(78),}$$

$$\frac{\omega_2}{k^2} + \frac{2}{3}\left\{\psi_2(\theta) + \frac{8}{\theta}\psi_2'(\theta)\right\}\phi_2 = 0 \quad \text{.....(79),}$$

where $\theta = ka$ as before. Eliminating ω_2/ϕ_2, and substituting the values of ψ_1, ψ_2 from (7) or (8), we obtain, after some reduction,

$$\tan\theta = \frac{-5\theta^5 + 92\theta^3 - 480\theta}{\theta^6 - 25\theta^4 + 252\theta^2 - 480} \quad \text{.....(80).}$$

Solving this, I find

$$ka/\pi = \cdot 8485,\ 1\cdot 7420,\ 2\cdot 8257,\ 3\cdot 8709,\ 4\cdot 8974,\ 5\cdot 9148,\ \&c. \quad \text{....(81).}$$

The frequency of the first mode is therefore much lower than in the species $n = 1$.

The radii of the spherical surfaces of zero normal motion are to be found from

$$\psi_2(kr) = \Upsilon \quad \text{.....(82),}$$

where

$$\Upsilon = \frac{k^2a^2\psi_2(ka) - 10\psi_1(ka)}{k^2a^2 - 10} \quad \text{.....(83),†}$$

by (78).

The character of the vibration will in other respects depend upon the form of ϕ_2. For the case of the zonal harmonic we may utilize the formulæ (68). If we write $\phi_2 = 2x^2 - y^2 - z^2$, this becomes

$$\Psi = \tfrac{2}{3}x\varpi^2\{\psi_2(kr) - \Upsilon\} \quad \text{.....(84).}$$

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* For $s = 1$, I find $k\varpi/\pi = \cdot 792$, or $\varpi/a = \cdot 643$. For $s = 2$, $k\varpi/\pi = \cdot 885$ and $1\cdot 935$, whence $\varpi/a = \cdot 374$ and $\cdot 817$. The spheres on which the nodal circles are great circles cut the lines of motion at right angles.

† For the first and second roots, I find $\Upsilon = \cdot 1171$ and $\Upsilon = \cdot 0789$, respectively.

I have traced, in Figs. 3 and 4, the forms of the lines of motion in the first two modes. The first mode is, from one point of view, the most important of all the fundamental modes of the sphere, for it corresponds to the gravest longitudinal vibration of a bar, or other elongated body. It is interesting to trace in imagination the changes in the forms of the lines of motion as we pass by gradual stages from such a body to a sphere.

It may easily be shown that, in all the modes at present under consideration, the amplitudes of vibration at two points equidistant from the centre, and situate, one in the axis of the harmonic ϕ_2, the other in the equatorial plane, are in the ratio of two to one.

The nature of the vibrations when the harmonic ϕ_2 is sectorial may be examined by making $\phi_2 = y^2 - z^2$. In the gravest mode, the y-diameter expands whilst the z-diameter contracts, and *vice versa*, all points in the x-diameter remaining at rest.

The species of higher orders ($n = 3$, 4, &c.) seem hardly of sufficient interest to call for a detailed examination. The main difficulty, in attempting to form a clear idea of the nature of the various fundamental modes, arises from our want of familiarity with the nature and properties of the various types of spherical harmonic which are involved. When these are thoroughly comprehended, the statements made near the end of § 5 give a fair indication of the characters of the corresponding vibrations.

12. It remains to examine in what way the conclusions of § 11 must be modified when the substance of the sphere is not incompressible.

Species $n = 1$. $[\sigma < \frac{1}{2}.]$

The equations (29), (30) become

$$\psi_1(ha) \cdot \frac{\omega_1}{h^2} - \psi_1(ka) \cdot \phi_1 = 0 \quad \text{.....(85),*}$$

$$\left\{ \psi_1(ha) + \frac{6}{k^2a^2} ha\psi_1'(ha) \right\} \frac{\omega_1}{h^2} + \tfrac{1}{2} \left\{ \psi_1(ka) + \frac{6}{ka} \psi_1'(ka) \right\} \phi_1 = 0 \quad \text{...(86).}$$

Eliminating the ratio ω_1/ϕ_1, we obtain

$$1 + \frac{4\lambda\psi_1(\lambda\theta)}{\theta\psi_1(\lambda\theta)} + \frac{2\psi_1'(\theta)}{\theta\psi_1(\theta)} = 0 \quad \text{.....(87),}$$

where $\theta = ka$, and $\lambda = h/k$, $= \sqrt{\{\mathfrak{n}/(\mathfrak{m}+\mathfrak{n})\}}$, as before. It is readily found that the expression on the left-hand side of (87) is positive for small values of θ, and again that the fraction $\psi_1'(\theta)/\psi_1(\theta)$ changes

* See the footnote to equation (70).

from $-\infty$ to $+\infty$ whenever θ, increasing,* passes through a root of $\psi_1(\theta) = 0$. Hence, if $\vartheta_1, \vartheta_2, \vartheta_3$, &c., be the combined system of roots of the two equations $\psi_1(\theta) = 0$, $\psi_1(\lambda\theta) = 0$, or (say) the roots of

$$\psi_1(\theta) \cdot \psi_1(\lambda\theta) = 0,$$

arranged in ascending order of magnitude, it appears that one root of (87) lies between 0 and ϑ_1, a second between ϑ_1 and ϑ_2, a third between ϑ_2 and ϑ_3, and so on. It can thence be shown, by much the same method as in Todhunter's *Functions of Laplace, &c.*, § 396, that the roots of (87) are all real, and that only one root lies within each of the intervals specified. The actual calculation of the roots is laborious. A good approximation to the first root may be made by the method of Lord Rayleigh's paper already cited. The form of the method which I have adopted is that ascribed by Prof. Cayley (in a note at the end of the paper in question) to Encke. If the equation (87) be cleared of fractions, and expanded in rising powers of θ, it assumes, after division by θ^2, the form

$$P - Q\theta^2 + R\theta^4 - S\theta^6 + T\theta^8 - \&c. = 0 \quad \ldots\ldots\ldots\ldots\ldots (88).$$

I find, for the values of the coefficients,

$$P = 3 - 4\lambda^2,$$

$$Q = \frac{5}{14} - \frac{1}{10}\lambda^2 - \frac{2}{7}\lambda^4,$$

$$R = \frac{1}{72} + \frac{3}{140}\lambda^2 - \frac{1}{56}\lambda^4 - \frac{1}{126}\lambda^6,$$

$$S = \frac{1}{3696} + \frac{17}{15120}\lambda^2 + \frac{1}{3920}\lambda^4 - \frac{1}{1680}\lambda^6 - \frac{1}{8316}\lambda^8,$$

$$T = \frac{1}{314496} + \frac{1}{41580}\lambda^2 + \frac{13}{423360}\lambda^4 - \frac{1}{211680}\lambda^6 - \frac{13}{1330560}\lambda^8 - \frac{1}{864864}\lambda^{10}.$$

Now, let $a = Q/P$, $b = R/P$, $c = S/P$, &c., so that (88) may be written

$$1 - a\theta^2 + b\theta^4 - c\theta^6 + d\theta^8 - \&c. = 0 \quad \ldots\ldots\ldots\ldots\ldots (89).$$

If the equation whose roots are the squares of the roots of this be

$$1 - a_1\theta^2 + b_1\theta^4 - \&c. = 0,$$

we have $a_1 = a^2 - 2b$, $b_1 = b^2 - 2ac + 2d$, &c.,

and we may suppose a_2, b_2, &c. derived from a_1, b_1, &c., in like manner. I have calculated the value of a_2 for the cases

$$\lambda^2 = \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{7}, \tfrac{1}{4}, 0,$$

which correspond to

$$\sigma = 0, \tfrac{1}{4}, \tfrac{3}{10}, \tfrac{1}{3}, \tfrac{1}{2}.$$

* We are here considering only positive values of θ. The roots of (87) occur of course in pairs, of equal magnitude and opposite sign.

If θ_1, θ_2, θ_3, &c. denote the roots of (89), we have $a_2 = \Sigma_1^{\infty} \theta^{-8}$. By equating a_2 to θ_1^{-8}, we obtain an approximate value of θ_1.* To obtain a more accurate value, we may apply a correction for the higher roots (which need not be known very accurately) in the manner explained in the paper referred to. The numbers given in the first line of the following table were calculated in this way, and afterwards verified on the equation obtained from (87), by substituting the values of $\psi_1(\theta)$, $\psi_1(\lambda\theta)$ from (8). The table gives the values of ka/π for the first three roots in each of the above-mentioned cases. The values for the second and third roots were found by tentative methods, except in the case $\sigma = \frac{1}{2}$, where they are repeated from (74).

s	$\sigma = 0$	$\sigma = \frac{1}{4}$	$\sigma = \frac{3}{10}$	$\sigma = \frac{1}{3}$	$\sigma = \frac{1}{2}$
1	·900	1·090	1·124	1·145	1·232
2	1·770	2·155	2·260	2·305	2·369
3	2·387	2·465	2·571	2·721	3·410

This table is a further illustration of Lord Rayleigh's principle cited above under the head of the radial vibrations. It appears that the value of m has a very considerable influence on the frequency.

The radial component of the vibration is, by (34*a*),

$$-\frac{1}{h^2}\frac{d\delta_1}{dr} + \frac{1}{r}\psi_1(kr)\,\phi_1.$$

Writing $\phi_1 = x$, and making use of (34) and (85), we put this into the form

$$\left[\psi_1(kr) - \frac{\psi_1(ka)}{\psi_1'(ka)}\{\psi_1(hr) + hr\psi_1'(hr)\}\right]\frac{x}{r}.$$

The radii of the spherical surfaces of zero normal motion are the values of r which make this expression vanish. When $r=a$ the value of the expression within square brackets becomes

$$= -\frac{ha\psi_1'(ha)}{\psi_1(ha)}\psi_1(ka)$$

$$= \frac{ka}{4}\{ka\psi_1(ka) + 2\psi_1'(ka)\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(90),$$

the transformation being effected by means of (87). We verify, from either of these forms, that in the particular case $\sigma = \frac{1}{2}$ the radial motion is zero at the boundary. For other values of σ, there is in general a finite component of vibration in the direction of the normal. It is easily seen that the expression

$$\theta\psi_1(\theta) + 2\psi_1'(\theta)$$

* The values so obtained are in fact too small by about seven or eight units in the fourth place of decimals.

is positive for small values of θ; and the values of θ/π for which it changes sign have been given in (74). Hence, and from the foregoing table, we learn that (for values of $\sigma < \frac{1}{2}$) the signs of the expression (90) are, for the first three modes, +, −, +, respectively. We infer, with certainty as to the second mode, and with great probability as to the third, that for all values of σ there are in these cases one and two internal spherical surfaces, respectively, over which the radial motion is zero.* A rough idea of the arrangement of the lines of motion in these two modes may (at all events for values of σ only a little less than $\frac{1}{2}$) be obtained by drawing, in Figs. 1 and 2, a circle concentric with, and a little *inside*, the external boundary, and by taking the circle so drawn to represent the surface of the sphere.

Some further indications as to the influence of the value of σ on the character of the vibration may be obtained by comparing the signs of the vibration at the centre of the sphere and at points on the surface on the equator of the harmonic ϕ_1. The amplitude at the centre is

$$a_0 = 1 - \frac{\psi_1(ka)}{\psi_1(ha)},$$

and the amplitude at any point of the equator is

$$a_E = -\frac{k^2a^2}{10}\psi_2(ka).$$

It is readily found, for the first three modes, that a_0 is +, and that the signs of a_E are as given by the following scheme—

s	$\sigma = 0$	$\sigma = \frac{1}{4}$	$\sigma = \frac{3}{10}$	$\sigma = \frac{1}{3}$	$\sigma = \frac{1}{2}$
1	−	−	−	−	−
2	−	+	+	+	+
3	+	+	+	+	−

From this we gather that in the *first* mode there is always a nodal circle; that in the *second* mode there are two nodal circles if σ is not less than $\frac{1}{4}$, but that the second nodal circle disappears for some value of σ between 0 and $\frac{1}{4}$; and that in the *third* mode, although for $\sigma = \frac{1}{2}$ there are three nodal circles, yet the third of these disappears for some value of σ between $\frac{1}{3}$ and $\frac{1}{2}$.

Species $n = 2$. $[\sigma < \frac{1}{2}.]$

The equations (29), (30) become, when $n = 2$,

$$\left\{\frac{k^2a^2}{5}\psi_2(ha) - 2\psi_1(ha)\right\}\frac{\omega_2}{k^2} - \left\{\frac{k^3a^2}{5}\psi_2(ka) - 2\psi_1(ka)\right\}\phi_2 = 0 \ldots (91),$$

* We cannot, however, generalize this statement, and assert that in the s^{th} mode there are always $s-1$ such surfaces. There will in general be less than this number.

$$\left\{\psi_2(ha)+\frac{8}{h^2a^2}ha\psi_2'(ha)\right\}\frac{\omega_2}{h^2}+\tfrac{2}{3}\left\{\psi_2(ka)+\frac{8}{k^2a^2}ka\psi_2'(ka)\right\}\phi_2=0 \quad ..(92).$$

The equation derived by eliminating ω_2/ϕ_2 is considerably more complicated than in the former species. I have therefore confined myself to the calculation of the lowest root, by Encke's method, for the same values of σ as before. When cleared of fractions and expanded in rising powers of θ, the equation in question assumes the form

$$P-Q\theta^2+R\theta^4-S\theta^6+T\theta^8-\&c.=0,$$

where $\theta=ka$. I find, for the values of the coefficients,

$$P=19-24\lambda^2,$$

$$Q=\frac{37}{9}-\frac{7}{2}\lambda^2-\frac{4}{3}\lambda^4,$$

$$R=\frac{193}{792}+\frac{5}{126}\lambda^2-\frac{13}{56}\lambda^4-\frac{1}{33}\lambda^6,$$

$$S=\frac{25}{3861}+\frac{125}{11088}\lambda^2-\frac{1}{168}\lambda^4-\frac{185}{33264}\lambda^6-\frac{1}{2574}\lambda^8,$$

$$T=\frac{719}{7413120}+\frac{163}{432432}\lambda^2+\frac{19}{133056}\lambda^4-\frac{59}{299376}\lambda^6-\frac{253}{3459456}\lambda^8-\frac{1}{308880}\lambda^{10},$$

where λ^2 has the same meaning as before. When the value of a_2 for any assigned value λ^2 has been computed, the correction for the higher roots may be roughly estimated from its known value in the case $\lambda^2=0$. In this way I have obtained the following values of ka/π, which are probably accurate to the last figure given:—

$\sigma=0$	$\sigma=\frac{1}{4}$	$\sigma=\frac{3}{10}$	$\sigma=\frac{1}{3}$	$\sigma=\frac{1}{2}$
·823	·840	·842	·843	·848

It appears that the influence of the value of m on the frequency is (especially within such limits of the ratio m/n as include most actual solids) very slight.

13. As an application of the preceding results we may calculate the frequency of vibration of a steel ball one centimetre in radius, for the slowest of those fundamental modes in which the surface oscillates in the form of a harmonic spheroid of the second order. In § 12 we obtained for this case $ka/\pi=\cdot842$. Now, $ka/\pi=T_0/\tau$, where τ is the period, and $T_0=2a/\sqrt{(n\rho^{-1})}$. Making then $a=1$, and adopting

from Everett* the values $n = 8{\cdot}19 \times 10^{11}$, $\rho = 7{\cdot}85$, in C. G. S. measure, I find that the frequency $\tau^{-1} = 136000$, about. For a steel globe of any other dimensions, this result must be divided by the radius in centimetres. For a globe of the size of the earth $[a = 6{\cdot}37 \times 10^8]$, I find that the period $\tau = 1$ hr. 18 m.†

On some Formulæ arising from the Differentiation of Elliptic Functions with regard to the Modulus. *By* Rev. M. M. U. Wilkinson.

[*Read June 8th*, 1882.]

1. If we take the identity,

$$2\,\mathrm{cn}(\alpha-\beta)\,\mathrm{cn}(\beta-\gamma)\,\mathrm{cn}(\gamma-\alpha) = 2-\mathrm{sn}^2(\alpha-\beta)-\mathrm{sn}^2(\beta-\gamma)-\mathrm{sn}^2(\gamma-\alpha) + k^2\,\mathrm{sn}^2(\alpha-\beta)\,\mathrm{sn}^2(\beta-\gamma)\,\mathrm{sn}^2(\gamma-\alpha),$$

and differentiate with respect to k, we have

$$\begin{aligned} &2k\,\mathrm{sn}^2(\alpha-\beta)\,\mathrm{sn}^2(\beta-\gamma)\,\mathrm{sn}^2(\gamma-\alpha) \\ &= \{-2\,\mathrm{sn}(\alpha-\beta)\,\mathrm{cn}(\beta-\gamma)\,\mathrm{cn}(\gamma-\alpha)+2\,\mathrm{sn}(\alpha-\beta)\,\mathrm{cn}(\alpha-\beta) \\ &\qquad -2k^3\,\mathrm{sn}(\alpha-\beta)\,\mathrm{cn}(\alpha-\beta)\,\mathrm{sn}^2(\beta-\gamma)\,\mathrm{sn}^2(\gamma-\alpha)\}\frac{d\,.\,\mathrm{am}(\alpha-\beta)}{dk} \\ &= 2\{-\mathrm{dn}(\alpha-\beta)-k^2\,\mathrm{sn}(\beta-\gamma)\,\mathrm{sn}(\gamma-\alpha)\} \\ &\qquad \times \mathrm{sn}(\alpha-\beta)\,\mathrm{sn}(\beta-\gamma)\,\mathrm{sn}(\gamma-\alpha)\frac{d\,.\,\mathrm{am}(\alpha-\beta)}{dk} \\ &= -2\,\mathrm{dn}(\beta-\gamma)\,\mathrm{dn}(\gamma-\alpha)\,\mathrm{sn}(\alpha-\beta)\,\mathrm{sn}(\beta-\gamma)\,\mathrm{sn}(\gamma-\alpha)\frac{d\,.\,\mathrm{am}(\alpha-\beta)}{dk}; \end{aligned}$$

whence

$$\frac{1}{\mathrm{dn}(\alpha-\beta)}\frac{d\,.\,\mathrm{am}(\alpha-\beta)}{dk}+\frac{1}{\mathrm{dn}(\beta-\gamma)}\frac{d\,.\,\mathrm{am}(\beta-\gamma)}{dk}+\frac{1}{\mathrm{dn}(\gamma-\alpha)}\frac{d\,.\,\mathrm{am}(\gamma-\alpha)}{dk} = -\frac{k\,\mathrm{sn}(\alpha-\beta)\,\mathrm{sn}(\beta-\gamma)\,\mathrm{sn}(\gamma-\alpha)}{\mathrm{dn}(\alpha-\beta)\,\mathrm{dn}(\beta-\gamma)\,\mathrm{dn}(\gamma-\alpha)}.$$

2. It follows from this, that if

$$x_1+x_2+x_3+x_4 = 0,$$

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* *Units and Physical Constants*, p. 53.

† *Cf.* Thomson, *Phil. Trans.*, 1863, p. 573.

we have

$$\frac{1}{\operatorname{dn} x_1}\frac{d.\operatorname{am} x_1}{dk}+\frac{1}{\operatorname{dn} x_2}\frac{d.\operatorname{am} x_2}{dk}+\frac{1}{\operatorname{dn} x_3}\frac{d.\operatorname{am} x_3}{dk}+\frac{1}{\operatorname{dn} x_4}\frac{d.\operatorname{am} x_4}{dk}$$

$$=\frac{k\operatorname{sn} x_1\operatorname{sn} x_2\operatorname{sn}(x_1+x_2)}{\operatorname{dn} x_1\operatorname{dn} x_2\operatorname{dn}(x_1+x_2)}+\frac{k\operatorname{sn} x_3\operatorname{sn} x_4\operatorname{sn}(x_3+x_4)}{\operatorname{dn} x_3\operatorname{dn} x_4\operatorname{dn}(x_3+x_4)}$$

$$=\frac{k\operatorname{sn}(x_1+x_2)}{\operatorname{dn}(x_1+x_2)}\left\{\frac{\operatorname{sn} x_1\operatorname{sn} x_2}{\operatorname{dn} x_1\operatorname{dn} x_2}-\frac{\operatorname{sn} x_3\operatorname{sn} x_4}{\operatorname{dn} x_3\operatorname{dn} x_4}\right\}$$

$$=\frac{k\operatorname{sn}(x_1+x_2)(\operatorname{cn} x_1\operatorname{cn} x_2-\operatorname{cn} x_3\operatorname{cn} x_4)}{\operatorname{dn}(x_1+x_2)\operatorname{dn} x_1\operatorname{dn} x_2\operatorname{dn} x_3\operatorname{dn} x_4}$$

$$=\frac{k\operatorname{sn}(x_1+x_2)\{\operatorname{sn} x_1\operatorname{sn} x_2\operatorname{dn}(x_1+x_2)-\operatorname{sn} x_3\operatorname{sn} x_4\operatorname{dn}(x_1+x_2)\}}{\operatorname{dn}(x_1+x_2)\operatorname{dn} x_1\operatorname{dn} x_2\operatorname{dn} x_3\operatorname{dn} x_4}$$

$$=\frac{k\operatorname{sn}(x_1+x_2)(\operatorname{sn} x_1\operatorname{sn} x_2-\operatorname{sn} x_3\operatorname{sn} x_4)}{\operatorname{dn} x_1\operatorname{dn} x_2\operatorname{dn} x_3\operatorname{dn} x_4}$$

$$=\frac{k\operatorname{sn}(x_1+x_2)\operatorname{sn}(x_1+x_3)\operatorname{sn}(x_2+x_3)(1-k^2\operatorname{sn} x_1\operatorname{sn} x_2\operatorname{sn} x_3\operatorname{sn} x_4)}{\operatorname{dn} x_1\operatorname{dn} x_2\operatorname{dn} x_3\operatorname{dn} x_4}$$

$$=-\frac{k\operatorname{sn}(x_1+x_2)\operatorname{sn}(x_1+x_3)\operatorname{sn}(x_1+x_4)(1-k^2\operatorname{sn} x_1\operatorname{sn} x_2\operatorname{sn} x_3\operatorname{sn} x_4)}{\operatorname{dn} x_1\operatorname{dn} x_2\operatorname{dn} x_3\operatorname{dn} x_4}.$$

3. In the case of $x_1+x_2+x_3+x_4+x_5=0$,

we shall require the following formulæ, where

$$u+v+r+s=0,$$

$$k^2k'^2\operatorname{sn} u\operatorname{sn} v\operatorname{sn} r\operatorname{sn} s+\operatorname{dn} u\operatorname{dn} v\operatorname{dn} r\operatorname{dn} s=k'^2+k^2\operatorname{cn} u\operatorname{cn} v\operatorname{cn} r\operatorname{cn} s\ldots(1);$$

$$\operatorname{cn} u\operatorname{cn} v\operatorname{dn} r\operatorname{dn} s-\operatorname{cn} r\operatorname{cn} s\operatorname{dn} u\operatorname{dn} v=k'^2(\operatorname{sn} u\operatorname{sn} v-\operatorname{sn} r\operatorname{sn} s)\ldots(2);$$

$$k^2\operatorname{sn}(u+v)(\operatorname{sn} u\operatorname{sn} v-\operatorname{sn} r\operatorname{sn} s)=Zu+Zv+Zr+Zs\ldots\ldots(3);$$

$$k'^2\operatorname{sn} u\operatorname{sn} v\operatorname{sn} r\operatorname{cn} s\operatorname{dn} s-\operatorname{dn} u\operatorname{dn} v\operatorname{dn} r\operatorname{cn} s\operatorname{sn} s$$
$$+\operatorname{cn} u\operatorname{cn} v\operatorname{cn} r\operatorname{sn} s\operatorname{dn} s=k^2\operatorname{sn}(r+s)(\operatorname{sn} u\operatorname{sn} v-\operatorname{sn} r\operatorname{sn} s)\ldots(4);$$

$$k^2\operatorname{sn} u\operatorname{sn} v\operatorname{sn} r\operatorname{dn} s+\operatorname{dn} u\operatorname{dn} v\operatorname{dn} r\operatorname{sn} s$$
$$=\operatorname{sn} s\operatorname{dn} s+k^2\operatorname{cn} s\operatorname{sn}(r+s)(\operatorname{sn} u\operatorname{sn} v-\operatorname{sn} r\operatorname{sn} s)\ldots(5).$$

The first two are given by Prof. Cayley and Prof. H. J. S. Smith, in *Proceedings* for January and February, 1879 (see p. 97). (3) is obvious.

To establish (4), we have, considering u, v as constant in (1), differentiating, and dividing by k^2,

$$k'^2\operatorname{sn} u\operatorname{sn} v\operatorname{sn} r\operatorname{cn} s\operatorname{dn} s$$
$$-\operatorname{dn} u\operatorname{dn} v\operatorname{dn} r\operatorname{cn} s\operatorname{sn} s$$
$$+\operatorname{cn} u\operatorname{cn} v\operatorname{cn} r\operatorname{sn} s\operatorname{dn} s=A$$

$$=k'^2\operatorname{sn} u\operatorname{sn} v\operatorname{sn} s\operatorname{cn} r\operatorname{dn} r$$
$$-\operatorname{dn} u\operatorname{dn} v\operatorname{dn} s\operatorname{cn} r\operatorname{sn} r$$
$$+\operatorname{cn} u\operatorname{cn} v\operatorname{cn} s\operatorname{sn} r\operatorname{dn} r;$$

$$\begin{aligned}2A &= k'^2 \operatorname{sn} u \operatorname{sn} v (\operatorname{sn} r \operatorname{cn} s \operatorname{dn} s + \operatorname{sn} s \operatorname{cn} r \operatorname{dn} r)\\ &\quad - \operatorname{dn} u \operatorname{dn} v (\operatorname{dn} r \operatorname{cn} s \operatorname{sn} s + \operatorname{dn} s \operatorname{cn} r \operatorname{sn} r)\\ &\quad + \operatorname{cn} u \operatorname{cn} v (\operatorname{cn} r \operatorname{sn} s \operatorname{dn} s + \operatorname{cn} s \operatorname{sn} r \operatorname{dn} r)\\ &= \operatorname{sn}(r+s)(1-k^2 \operatorname{sn}^2 r \operatorname{sn}^2 s)\{k'^2 \operatorname{sn} u \operatorname{sn} v - \operatorname{dn} u \operatorname{dn} v \operatorname{cn}(r-s)\\ &\qquad + \operatorname{cn} u \operatorname{cn} v \operatorname{dn}(r-s)\}\\ &= k'^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn}(r+s)(1-k^2 \operatorname{sn}^2 r \operatorname{sn}^2 s)\\ &\quad + \operatorname{sn}(r+s)\{-\operatorname{dn} u \operatorname{dn} v \operatorname{cn} r \operatorname{cn} s + \operatorname{cn} u \operatorname{cn} v \operatorname{dn} r \operatorname{dn} s\\ &\qquad - \operatorname{sn} r \operatorname{sn} s (\operatorname{dn} u \operatorname{dn} v \operatorname{dn} r \operatorname{dn} s - k^2 \operatorname{cn} u \operatorname{cn} v \operatorname{cn} r \operatorname{cn} s)\}\\ &= k'^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn}(r+s)(1-k^2 \operatorname{sn}^2 r \operatorname{sn}^2 s)\\ &\quad + \operatorname{sn}(r+s)\{k'^2(\operatorname{sn} u \operatorname{sn} v - \operatorname{sn} r \operatorname{sn} s) - k'^2 \operatorname{sn} r \operatorname{sn} s + k^2 k'^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn}^2 r \operatorname{sn}^2 s\}\\ &= 2k'^2 \operatorname{sn}(r+s)(\operatorname{sn} u \operatorname{sn} v - \operatorname{sn} r \operatorname{sn} s).\end{aligned}$$

To establish (5), we get, from equations (4) and (1),

$$\begin{aligned}k'^2 \operatorname{dn} u \operatorname{dn} v \operatorname{dn} r \operatorname{sn}^3 s &= \operatorname{dn} u \operatorname{dn} v \operatorname{dn} r \operatorname{sn} s (\operatorname{dn}^2 s - \operatorname{cn}^2 s)\\ = k'^2 \operatorname{sn} s \operatorname{dn} s &+ k^2 \operatorname{cn} u \operatorname{cn} v \operatorname{cn} r \operatorname{cn} s \operatorname{sn} s \operatorname{dn} s - k^2 k'^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn} r \operatorname{sn}^2 s \operatorname{dn} s\\ &- k'^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn} r \operatorname{cn}^2 s \operatorname{dn} s - \operatorname{cn} u \operatorname{cn} v \operatorname{cn} r \operatorname{cn} s \operatorname{sn} s \operatorname{dn} s\\ &+ k'^2 \operatorname{cn} s \operatorname{sn}(r+s)(\operatorname{sn} u \operatorname{sn} v - \operatorname{sn} r \operatorname{sn} s);\end{aligned}$$

or, $$\begin{aligned}\operatorname{dn} u \operatorname{dn} u \operatorname{dn} r \operatorname{sn}^3 s &= \operatorname{sn} s \operatorname{dn} s - \operatorname{cn} u \operatorname{cn} v \operatorname{cn} r \operatorname{cn} s \operatorname{sn} s \operatorname{dn} s\\ - \operatorname{sn} u \operatorname{sn} v \operatorname{sn} r \operatorname{dn} s &(k^2 \operatorname{sn}^2 s + \operatorname{cn}^2 s) + \operatorname{cn} s \operatorname{sn}(r+s)(\operatorname{sn} u \operatorname{sn} v - \operatorname{sn} r \operatorname{sn} s);\end{aligned}$$

and $$\begin{aligned}&\operatorname{dn} u \operatorname{dn} v \operatorname{dn} r \operatorname{sn} s \operatorname{cn}^2 s\\ &= k'^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn} r \operatorname{cn}^2 s \operatorname{dn} s + \operatorname{cn} u \operatorname{cn} v \operatorname{cn} r \operatorname{cn} s \operatorname{sn} s \operatorname{dn} s\\ &\quad - k'^2 \operatorname{cn} s \operatorname{sn}(r+s)(\operatorname{sn} u \operatorname{sn} v - \operatorname{sn} r \operatorname{sn} s);\end{aligned}$$

so that $$\begin{aligned}\operatorname{dn} u \operatorname{dn} v \operatorname{dn} r \operatorname{sn} s &= \operatorname{sn} s \operatorname{dn} s - k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn} r \operatorname{dn} s\\ &\quad + k^2 \operatorname{cn} s \operatorname{sn}(r+s)(\operatorname{sn} u \operatorname{sn} v - \operatorname{sn} r \operatorname{sn} s).\end{aligned}$$

We have now to find simple expressions for F, when

$$x_1 + x_2 + x_3 + x_4 + x_5 = 0, \quad \text{or} \quad \Sigma x = 0,$$

and $$kF = \Pi \operatorname{dn} x \,.\, \Sigma \frac{1}{\operatorname{dn} x} \frac{d \,.\, \operatorname{am} x}{dk}.$$

Now, we have

$$\begin{aligned}&\frac{1}{\operatorname{dn} x_1} \frac{d \,.\, \operatorname{am} x_1}{dk} + \frac{1}{\operatorname{dn} x_2} \frac{d \,.\, \operatorname{am} x_2}{dk} + \frac{1}{\operatorname{dn} x_3} \frac{d \,.\, \operatorname{am} x_3}{dk} + \frac{1}{\operatorname{dn}(x_4+x_5)} \frac{d \,.\, \operatorname{am}(x_4+x_5)}{dk}\\ &\qquad = \frac{k \operatorname{sn}(x_1+x_2)\{\operatorname{sn} x_1 \operatorname{sn} x_2 - \operatorname{sn} x_3 \operatorname{sn}(x_4+x_5)\}}{\operatorname{dn} x_1 \operatorname{dn} x_2 \operatorname{dn} x_3 \operatorname{dn}(x_4+x_5)};\end{aligned}$$

$$\begin{aligned}&\frac{1}{\operatorname{dn} x_4} \frac{d \,.\, \operatorname{am} x_4}{dk} + \frac{1}{\operatorname{dn} x_5} \frac{d \,.\, \operatorname{am} x_5}{dk} - \frac{1}{\operatorname{dn}(x_4+x_5)} \frac{d \,.\, \operatorname{am}(x_4+x_5)}{dk}\\ &\qquad = \frac{k \operatorname{sn} x_4 \operatorname{sn} x_5 \operatorname{sn}(x_4+x_5)}{\operatorname{dn} x_4 \operatorname{dn} x_5 \operatorname{dn}(x_4+x_5)};\end{aligned}$$

whence

$$\begin{aligned}\operatorname{dn}(x_4+x_5)\,F &= \operatorname{dn}x_4\operatorname{dn}x_5\operatorname{sn}(x_1+x_2)\{\operatorname{sn}x_1\operatorname{sn}x_2-\operatorname{sn}x_3\operatorname{sn}(x_4+x_5)\}\\ &\quad+\operatorname{sn}x_4\operatorname{sn}x_5\operatorname{dn}x_1\operatorname{dn}x_2\operatorname{dn}x_3\operatorname{sn}(x_4+x_5)\\ &= \operatorname{dn}(x_4+x_5)\operatorname{sn}(x_1+x_2)\{\operatorname{sn}x_1\operatorname{sn}x_2-\operatorname{sn}x_3\operatorname{sn}(x_4+x_5)\}\\ &\quad+k^2\operatorname{sn}x_4\operatorname{sn}x_5\operatorname{cn}(x_4+x_5)\operatorname{sn}(x_1+x_2)\{\operatorname{sn}x_1\operatorname{sn}x_2-\operatorname{sn}x_3\operatorname{sn}(x_4+x_5)\}\\ &\quad+\operatorname{sn}x_4\operatorname{sn}x_5\operatorname{dn}x_1\operatorname{dn}x_2\operatorname{dn}x_3\operatorname{sn}(x_4+x_5)\\ &= \operatorname{dn}(x_4+x_5)\operatorname{sn}(x_1+x_2)\{\operatorname{sn}x_1\operatorname{sn}x_2-\operatorname{sn}x_3\operatorname{sn}(x_4+x_5)\}\\ &\quad+\operatorname{sn}x_4\operatorname{sn}x_5\{\operatorname{sn}(x_4+x_5)\operatorname{dn}(x_4+x_5)-k^2\operatorname{sn}x_1\operatorname{sn}x_2\operatorname{sn}x_3\operatorname{dn}(x_4+x_5)\},\end{aligned}$$

by virtue of equation (5). So that

$$\begin{aligned}F &= \operatorname{sn}(x_1+x_2)\operatorname{sn}x_1\operatorname{sn}x_2-\operatorname{sn}x_3\operatorname{sn}(x_1+x_2)\operatorname{sn}(x_4+x_5)\\ &\quad+\operatorname{sn}x_4\operatorname{sn}x_5\operatorname{sn}(x_4+x_5)-k^2\Pi\operatorname{sn}x\\ &= \frac{1}{k^2}\Sigma Zx-k^2\Pi\operatorname{sn}x.\end{aligned}$$

Another form for F is worthy of notice. We have

$$\begin{aligned}&\operatorname{sn}(x_1+x_2)[\operatorname{cn}(x_1+x_2)-\operatorname{cn}x_1\operatorname{cn}x_2]\operatorname{dn}x_3\operatorname{dn}x_4\operatorname{dn}x_5\\ &\quad-\operatorname{sn}(x_1+x_2)[\operatorname{dn}(x_1+x_2)-\operatorname{dn}x_1\operatorname{dn}x_2]\operatorname{cn}x_3\operatorname{cn}x_4\operatorname{cn}x_5\\ &= \operatorname{sn}(x_1+x_2)\operatorname{sn}x_1\operatorname{sn}x_2[k^2\operatorname{cn}(x_1+x_2)\operatorname{cn}x_3\operatorname{cn}x_4\operatorname{cn}x_5\\ &\quad-\operatorname{dn}(x_1+x_2)\operatorname{dn}x_3\operatorname{dn}x_4\operatorname{dn}x_5]\\ &= k'^2\operatorname{sn}(x_1+x_2)\operatorname{sn}x_1\operatorname{sn}x_2[k^2\operatorname{sn}(x_1+x_2)\operatorname{sn}x_3\operatorname{sn}x_4\operatorname{sn}x_5-1];\end{aligned}$$

and

$$\begin{aligned}&k'^2\operatorname{sn}x_3\operatorname{sn}x_4\operatorname{sn}x_5\operatorname{cn}(x_1+x_2)\operatorname{dn}x_1\operatorname{dn}x_2\\ &= k'^2\operatorname{sn}x_3\operatorname{sn}x_4\operatorname{sn}x_5\operatorname{cn}(x_1+x_2)\operatorname{dn}(x_1+x_2)\\ &\quad+k^2k'^2\operatorname{sn}x_1\operatorname{sn}x_2\operatorname{sn}x_3\operatorname{sn}x_4\operatorname{sn}x_5\operatorname{cn}^2(x_1+x_2);\end{aligned}$$

so that

$$\begin{aligned}&\operatorname{sn}(x_1+x_2)\operatorname{cn}x_1\operatorname{cn}x_2\operatorname{dn}x_3\operatorname{dn}x_4\operatorname{dn}x_5\\ &-\operatorname{sn}(x_1+x_2)\operatorname{dn}x_1\operatorname{dn}x_3\operatorname{cn}x_3\operatorname{cn}x_4\operatorname{cn}x_5\\ &-k'^2\operatorname{cn}(x_1+x_2)\operatorname{dn}x_1\operatorname{dn}x_2\operatorname{sn}x_3\operatorname{sn}x_4\operatorname{sn}x_5\\ &= \operatorname{sn}(x_1+x_2)\{\operatorname{cn}(x_1+x_2)\operatorname{dn}x_3\operatorname{dn}x_4\operatorname{dn}x_5-\operatorname{dn}(x_1+x_2)\operatorname{cn}x_3\operatorname{cn}x_4\operatorname{cn}x_5\}\\ &\quad+k'^2\operatorname{sn}(x_1+x_2)\operatorname{sn}x_1\operatorname{sn}x_2-k'^2\operatorname{sn}x_3\operatorname{sn}x_4\operatorname{sn}x_5\operatorname{cn}(x_1+x_2)\operatorname{dn}(x_1+x_2)\\ &\quad-k^2k'^2\Pi\operatorname{sn}x = k'^2\operatorname{sn}(x_1+x_2)\operatorname{sn}x_1\operatorname{sn}x_2\\ &\quad+k'^2\operatorname{sn}x_4\operatorname{sn}x_5\operatorname{sn}(x_4+x_5)+k'^2\operatorname{sn}(x_1+x_2+x_3)\operatorname{sn}(x_1+x_2)\operatorname{sn}x_3\\ &\quad-k^2k'^2\Pi\operatorname{sn}x = k'^2F.\end{aligned}$$

On the Invariants of a Certain Orthogonal Transformation, with special reference to the Theory of the Strains and Stresses of an Elastic Solid. By W. J. C. Sharp, M.A.

[*Read Jan. 12th*, 1882.]

I. If a, b, c, f, g, h are transformed according to the same law as x^2, y^2, z^2, yz, zx, xy; x, y, and z being the rectangular Cartesian coordinates of a point, transformed without change of origin; they will have a system of invariants entirely unaffected by the transformation—viz.,

$$\text{(i.)}\quad a+b+c \equiv D \text{ say,}$$

$$\text{(ii.)}\quad ab+bc+ca-f^2-g^2-h^2 \equiv S,$$

$$\text{(iii.)}\quad abc+2fgh-af^2-bg^2-ch^2 \equiv \Delta.$$

This is easily proved from the equations of transformation; for, since there are six equations of condition connecting the direction cosines, and six more expressing the new values of a, b, c, &c. in terms of the old, it is evident that there ought to be three resultants; whilst, from the symmetry of the relations which express the direct and inverse transformations, each of these must be a symmetrical function of a, b, c, &c., and a', b', c', &c.; and, as these relations will usually express some property of the system, independent of the particular axes, these symmetrical relations will usually be reducible to the form $\phi(a, b, c \ldots) = \phi(a', b', c' \ldots)$, *i.e.*, to invariants.

From the number of equations it appears that $(a, b, c \ldots \mathcal{X} x^2, y^2 \ldots) \equiv U$ ought to have two independent covariants.

The two simplest of these are

$$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy \equiv U',$$

where A, B, &c. are the first minors of the determinant

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix},$$

which is the polar reciprocal of U, and is the S mentioned in Dr. Salmon's *Geometry of Three Dimensions*, p. 44; and $x^2+y^2+z^2$ itself, or, perhaps better, the covariant F of this and U, viz.,

$$(B+C)\,x^2+(C+A)\,y^2+(A+B)\,z^2-2Fyz-2Gzx-2Hxy \equiv U'',$$

which satisfies the equation

$$U'' \equiv D(x^2+y^2+z^2)-U',$$

and may be derived from U' as Poinsot's *Momental Ellipsoid* is from

$$\Sigma mx^2 . \xi^2 + \ldots + 2\Sigma myz . \eta\zeta \ldots = c,$$

or the strain ellipsoid from the elongation quadric.

The relation between U and U' is that between two conjugate cones, or that between the momental ellipsoid and that of gyration, or that between the two expressions for the work of an electric current (Clerk-Maxwell's *Electricity and Magnetism*, Vol. I., p. 385) when the conditions there mentioned are fulfilled.

II. If u, v, w and u', v', w' be any two sets of quantities cogredient with xyz, the equations $\frac{u}{u'} = \frac{v}{v'} = \frac{w}{w'}$, which express the parallelism of the vectors $ui+vj+wk$ and $u'i+v'j+w'k$, will be unaltered by transformation.

The following are some examples of equations of the kind:—

$$\dagger\ \frac{x-x_1}{x_1-x_2} = \frac{y-y_1}{y_1-y_2} = \frac{z-z_1}{z_1-z_2},$$

the equations to the right line through (x_1, y_1, z_1) (x_2, y_2, z_2).

$$\dagger\ \frac{x-\xi}{\frac{dx}{ds}} = \frac{y-\eta}{\frac{dy}{ds}} = \frac{z-\zeta}{\frac{dz}{ds}},$$

the equations to the tangent to a curve in space at (x, y, z).

$$\dagger\ \frac{x-\xi}{\frac{dU}{dx}} = \frac{y-\eta}{\frac{dU}{dy}} = \frac{z-\zeta}{\frac{dU}{dz}},$$

those to the normal to the surface $U = 0$ at (x, y, z).

$$\dagger\ \frac{\frac{d^2x}{ds^2}}{\frac{dU}{dx}} = \frac{\frac{d^2y}{ds^2}}{\frac{dU}{dy}} = \frac{\frac{d^2z}{ds^2}}{\frac{dU}{dz}},$$

the differential equations to geodesics on $U = 0$.

$$\frac{\xi - \frac{N\Sigma Y - M\Sigma Z}{R^2}}{\Sigma X} = \frac{\eta - \frac{L\Sigma Z - N\Sigma X}{R^2}}{\Sigma Y} = \frac{\zeta - \frac{M\Sigma X - L\Sigma Y}{R^2}}{\Sigma Z},$$

the equations to the central axis in Statics.

$$\dagger\ \frac{dx}{\frac{dZ}{dy} - \frac{dY}{dz}} = \frac{dy}{\frac{dX}{dz} - \frac{dZ}{dx}} = \frac{dz}{\frac{dY}{dx} - \frac{dX}{dy}},$$

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† Quantities marked † are also unaltered by change of origin.

the differential equations to the curve of intersection of the surfaces of equal pressure and of equal density in Hydrostatics.

$$\dagger \frac{d\xi}{u} = \frac{d\eta}{v} = \frac{d\zeta}{w},$$

those to the lines of motion in Hydrokinetics; and

$$\frac{x}{\omega_1} = \frac{y}{\omega_2} = \frac{z}{\omega_3},$$

where ω_1, ω_2, ω_3 are the angular velocities about the axes, the equations to the instantaneous axis when a rigid body is acted upon by no forces.

III. Two special classes of functions of the kind denoted by a, b, c &c., naturally suggest themselves first for consideration.

Those in which a, b, &c., are proportional to u^2, v^2, w^2, vw, wu, uv; u, v, and w being cogredient with x, y, and z respectively. In this case the related quadric represents a pair of parallel planes, and the invariants S and Δ vanish identically, while D becomes $u^2+v^2+w^2$, the square of the tensor of the vector $ui+vj+wk$.

The following are some examples:—

$$x^2+y^2+z^2,$$

the square of the distance of (x, y, z) from the origin.

$$\dagger (x-x_1)^2+(y-y_1)^2+(z-z_1)^2,$$

that of the distance between (x, y, z) and (x_1, y_1, z_1).

$$\dagger (dx)^2+(dy)^2+(dz)^2,$$

the square of the element of the arc of a curve.

$$\dagger \left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2+\left(\frac{d^2z}{ds}\right),$$

the square of the curvature.

$$\dagger \left(\frac{dy}{dt}\cdot\frac{d^2z}{dt^2}-\frac{dz}{dt}\cdot\frac{d^2y}{dt^2}\right)^2+\left(\frac{dz}{dt}\cdot\frac{d^2x}{dt^2}-\frac{dx}{dt}\cdot\frac{d^2z}{dt^2}\right)^2+\left(\frac{dx}{dt}\cdot\frac{d^2y}{dt^2}-\frac{dy}{dt}\cdot\frac{d^2x}{dt^2}\right)^2,$$

an important expression in Solid Geometry.

$$\dagger X^2+Y^2+Z^2,$$

the square of the form of which X, Y, and Z are the components.

$$\dagger (\Sigma X)^2+(\Sigma Y)^2+(\Sigma Z)^2,$$

that of the resultant of a set of forces.

$$(Yz-Zy)^2+(Zx-Xz)^2+(Xy-Yx)^2,$$

or more generally

$$\{\Sigma(Yz-Zy)\}^2+\{\Sigma(Zx-Xz)\}^2+\{\Sigma(Xy-Yx)\}^2,$$

the square of the resultant moment.

the velocity squared. $$\dagger\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2,$$

the *vis viva*. $$\dagger\,\Sigma\frac{m}{2}\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\},$$

$$\dagger\left(\frac{d^2x}{dt^2}\right)^2+\left(\frac{d^2y}{dt^2}\right)^2+\left(\frac{d^2z}{dt^2}\right)^2,$$

the square of the resultant acceleration.

$$(\omega_1)^2+(\omega_2)^2+(\omega_3)^2,$$

that of the resultant angular velocity.

IV. Then the case in which a, b, c, &c. are proportional to uu', vv', ww', $\frac{1}{2}(vw'+v'w)$, $\frac{1}{2}(wu'+w'u)$, and $\frac{1}{2}(uv'+u'v)$; u, v, w and u', v', w' being cogredient to x, y, z. In this case the auxiliary quadric

$$(ux+vy+wz)(u'x+v'y+w'z)=c$$

is a cylinder. The invariant Δ vanishes identically, D takes the form of $uu'+vv'+ww'$, and S that of $(vw'-v'w)^2+(wu'-w'u)^2+(uv'-u'v)^2$.

If u, v, w and u', v', w' be the components of the vectors α, β, the last two invariants are $-S.\alpha\beta$ and $T^2.V\alpha\beta$.

If $v_1w_2-v_2w_1$, $w_1u_2-w_2u_1$, $u_1v_2-u_2v_1$ be substituted for u', v', and w' in the value of D, this takes the important form

$$\begin{vmatrix} u, & v, & w \\ u_1, & v_1, & w_1 \\ u_2, & v_2, & w_2 \end{vmatrix}^{*} \text{ or } -S.\alpha\beta\gamma,$$

if $$\alpha=ui+vj+wk,\quad \beta=u_1i+v_1j+w_1k,$$

and $$\gamma=u_2i+v_2j+w_2k;$$

and hence the Jacobian of any three functions of x, y, and z is an invariant.

The following are examples of these forms:—

$$xx'+yy'+zz',$$

or $rr'\cos\theta$, where r and r' are the distances of (x, y, z) and (x', y', z') from the origin and θ the angle between them.

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* The invariance of this determinant may also be easily proved by squaring.

$$(yz'-z'x)^2+(zx'-z'x)^2+(xy'-x'y)^2,$$

or $r^2r'^2\sin\theta$, four times the square of the area of the triangle of which these are sides.

$$\dagger \begin{vmatrix} x_1-x, & y_1-y, & z_1-z \\ x_2-x, & y_2-y, & z_2-z \\ x_3-x, & y_3-y, & z_3-z \end{vmatrix},$$

six times the volume of the tetrahedron whose vertices are (x, y, z) (x_1, y_1, z_1), &c.,—the quantity which, equated to zero, gives the equation to the plane through the three points (x_1, y_1, z_1), &c.

$$\dagger (x-\xi)\frac{dU}{dx}+(y-\eta)\frac{dU}{dy}+(z-\zeta)\frac{dU}{dz},$$

the left-hand side of the equation to the tangent to the surface $U=0$ at (x, y, z).

$$\dagger (x-\xi)\frac{ds}{dx}+(y-\eta)\frac{ds}{dy}+(z-\zeta)\frac{ds}{dz},$$

that of the equation to the normal plane to a curve at (x, y, z);

$$\dagger (dy\,.\,d^2z-dz\,.\,d^2y)(x-\xi)+(dz\,.\,d^2x-dx\,.\,d^2z)(y-\eta)$$
$$+(dx\,.\,d^2y-dy\,.\,d^2x)(z-\zeta),$$

that of the equation to the osculating plane.

$$\begin{vmatrix} x, & y, & z \\ U_1, & U_2, & U_3 \\ U_1', & U_2', & U_3' \end{vmatrix}=0,$$

the expression which is put to give the equation to the principal planes of the quadric $U=0$ (Salmon, *Geometry of Three Dimensions*, p. 44).

$$\begin{vmatrix} dx, & dy, & dz \\ \frac{dU}{dx}, & \frac{dU}{dy}, & \frac{dU}{dz} \\ d\frac{dU}{dx}, & d\frac{dU}{dy}, & d\frac{dU}{dz} \end{vmatrix},$$

the left-hand side of the differential equation to the lines of curvature upon the surface $U=0$.

The reciprocal of the radius of torsion

$$\dagger k^2\cdot\begin{vmatrix} \frac{dx}{dt}, & \frac{dy}{dt}, & \frac{dz}{dt} \\ \frac{d^2x}{dt^2}, & \frac{d^2y}{dt^2}, & \frac{d^2z}{dt^2} \\ \frac{d^3x}{dt^3}, & \frac{d^3y}{dt^3}, & \frac{d^3z}{dt^3} \end{vmatrix},$$

where
$$\frac{1}{k^2} = \left(\frac{dy}{dt}\cdot\frac{d^2z}{dt^2} - \frac{dz}{dt}\cdot\frac{d^2y}{dt^2}\right)^2$$
$$+\left(\frac{dz}{dt}\cdot\frac{d^2x}{dt^2} - \frac{dx}{dt}\cdot\frac{d^2z}{dt^2}\right)^2 + \left(\frac{dx}{dt}\cdot\frac{d^2y}{dt^2} - \frac{dy}{dt}\cdot\frac{d^2x}{dt^2}\right)^2,$$

which is itself an invariant.

$$\frac{\frac{dU}{dx}\cdot\frac{dV}{dx} + \frac{dU}{dy}\cdot\frac{dV}{dy} + \frac{dU}{dz}\cdot\frac{dV}{dz}}{\sqrt{\left\{\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2 + \left(\frac{dU}{dz}\right)^2\right\}}\sqrt{\left\{\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2\right\}}},$$

the cosine of the angle at which the two surfaces $U=0$, $V=0$ intersect.

$$\begin{vmatrix} U_1, & U_2, & U_3 \\ u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \end{vmatrix},$$

the left-hand side of the partial differential equation to a family of surfaces (Salmon's *Geometry of Three Dimensions*, p. 340). [If X, Y, Z be the components of a force and (x, y, z) its point of application.]

$$Xx + Yy + Zz,$$

the work done against the force from the origin to the point of application.

Its variation, $X\,.\,\delta x + Y\,.\,\delta y + Z\,.\,\delta z,$

the virtual work.

$$\int_a^b (X\,dx + Y\,dy + Z\,dz),$$

the work from A to B.

The same expressions with accelerations

$$\left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right)$$

substituted for the forces X, Y, Z.

The expression
$$\begin{vmatrix} X, & Y, & Z \\ \frac{dx}{dt}, & \frac{dy}{dt}, & \frac{dz}{dt} \\ \frac{d^2x}{dt^2}, & \frac{d^2y}{dt^2}, & \frac{d^2z}{dt^2} \end{vmatrix},$$

in the theory of inextensible strings.

$$\int m\left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right)ds,$$

and
$$\left\{\left(\int mX\,ds\right)^2 + \left(\int mY\,ds\right)^2 + \left(\int mZ\,ds\right)^2\right\}^{\frac{1}{2}},$$

two expressions for the tension; and

$$\begin{vmatrix} T\frac{d^2x}{ds^2}+mX, & \frac{dx}{ds}, & \frac{du}{dx} \\ T\frac{d^2y}{ds^2}+mY, & \frac{dy}{ds}, & \frac{du}{dy} \\ T\frac{d^2z}{ds^2}+mZ, & \frac{dz}{ds}, & \frac{du}{dz} \end{vmatrix}$$

(see Minchin's *Statics*, p. 351).

The quantity $u\omega_1+v\omega_2+w\omega_3$, the vanishing of which is a condition that the motion of a rigid body should be a mere rotation.

The functions

$$u\,dx+v\,dy+w\,dz,$$

$$u\frac{d\rho}{dx}+v\frac{d\rho}{dy}+w\frac{d\rho}{dz},$$

$$u\frac{dx}{ds}+v\frac{dy}{ds}+w\frac{dz}{ds},$$

in Hydrostatics.

The expression

$$\frac{dV}{dx}\cdot\frac{dx}{dh}+\frac{dV}{dy}\cdot\frac{dy}{dh}+\frac{dV}{dz}\cdot\frac{dz}{dh}$$

in the theory of the Potential (Clerk-Maxwell's *Electricity*, *&c.*, Vol. I., p. 180).

The numerous expressions for work in the same book. The quantity

$$\Pi=\frac{1}{r^3}\begin{vmatrix} \xi-x, & \eta-y, & \zeta-z \\ \frac{d\xi}{d\sigma}, & \frac{d\eta}{d\sigma}, & \frac{d\zeta}{d\sigma} \\ \frac{dx}{ds}, & \frac{dy}{ds}, & \frac{dz}{ds} \end{vmatrix}$$

(Vol. II., p. 39), and

$$X\left(\frac{dZ}{dy}-\frac{dY}{dz}\right)+Y\left(\frac{dX}{dz}-\frac{dZ}{dx}\right)+Z\left(\frac{dY}{dx}-\frac{dX}{dy}\right),$$

the vanishing of which is the condition that $X\,dx+Y\,dy+Z\,dz$ should become an exact differential when multiplied by a factor.

V. Turning now to the general case of functions which transform like a, b, c, &c. One very important set is that in which a, b, c, f, g, h are proportional to ΣXx, ΣYy, ΣZz, $\frac{1}{2}\Sigma(Yz+Zy)$, $\frac{1}{2}(Zx+Xz)$, $\frac{1}{2}\Sigma(Xy+Yx)$ respectively. X_1, X_2, &c., Y_1, Y_2, &c., Z_1, Z_2, &c., being cogredient with x, y, and z. In this case $\Sigma Xx+\Sigma Yy+\Sigma Zz$ is the in-

variant D; whilst

$$\{\Sigma(Yz-Zy)\}^2+\{\Sigma(Zx-Xz)\}^2+\{\Sigma(Xy-Yx)\}$$

is an invariant, by (III.)*

(i.) If X_1, Y_1, Z_1, &c. be forces, and (x_1, y_1, z_1) their points of application; $\Sigma(X\delta x+Y\delta y+Z\delta z)$,

the variation of $\Sigma(Xx+Yy+Zz)$, the virtual work;

$$(\Sigma X)^2+(\Sigma Y)^2+(\Sigma Z)^2,$$

the square of the resultant of forces equal to the given forces and all acting at one point;

$$\{\Sigma(Yz-Zy)\}^2+\{\Sigma(Zx-Xz)\}^2+\{\Sigma(Xy-Yx)\}^2,$$

the square of the moment of the couple produced by fixing the origin, —are all invariants.

$L.\Sigma X+M.\Sigma Y+N.\Sigma Z$ is also an invariant, and therefore

$$\frac{L.\Sigma X+M.\Sigma Y+N.\Sigma Z}{\sqrt{\{(\Sigma X)^2+(\Sigma Y)^2+(\Sigma Z)^2\}}}$$

the principal moment. The equations to the central axis are invariable (III.), and the pitch of the screw along this (Dr. Ball's *Theory of Screws*, p. 5) $= \dfrac{L.\Sigma X+M.\Sigma Y+N.\Sigma Z}{(\Sigma X)^2+(\Sigma Y)^2+(\Sigma Z)^2}$ also.

If this vanish, the system of forces is reducible to a single resultant.

(ii.) If accelerations $\dfrac{d^2x}{dt^2}$, $\dfrac{d^2y}{dt^2}$, $\dfrac{d^2z}{dt^2}$ be substituted for X, Y, Z, &c., the expressions $\Sigma m\left\{\dfrac{d^2x}{dt^2}\cdot\dfrac{dx}{dt}+\dfrac{d^2y}{dt^2}\cdot\dfrac{dy}{dt}+\dfrac{d^2z}{dt^2}\cdot\dfrac{dz}{dt}\right\}$

and $$\Sigma m\left(X\frac{dx}{dt}+Y\frac{dy}{dt}+Z\frac{dz}{dt}\right)$$

are invariants as well as those on both sides of the equation of *vis viva*,

$$\Sigma m\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\}=C+\Sigma m\int(Xdx+Ydy+Zdz),$$

which they serve to establish.

So also are $$\Sigma m\left(\frac{d^2x}{dt^2}\delta x+\frac{d^2y}{dt^2}\delta y+\frac{d^2z}{dt}\delta z\right),$$

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* This expression

$=-4S+4\{\Sigma Xx.\Sigma Yy+\Sigma Yy.\Sigma Zz+\Sigma Zz.\Sigma Xx-\Sigma Xy.\Sigma Yx-\Sigma Yz.\Sigma Zy-\Sigma Zx.\Sigma Xz\}$, so that the quantity in the bracket is also an invariant.

or $$\Sigma\left(\frac{d}{dt}\cdot\frac{dT}{d\theta'}-\frac{dT}{d\theta}\right)\tfrac{1}{2}\delta\theta,$$

and $$\Sigma m\,(X\delta x+X\delta y+Z\delta z),\text{ or }\delta U,$$

the expressions involved in Lagrange's equations.

The generalized components of force, acceleration, and momentum,

$$\Sigma m\left(X\frac{dx}{dq_r}+Y\frac{dy}{dq_r}+Z\frac{dz}{dq_r}\right),$$

$$\Sigma m\left(\frac{d^2x}{dt^2}\cdot\frac{dx}{dq_r}+\frac{d^2y}{dt^2}\cdot\frac{dy}{dq_r}+\frac{d^2z}{dt^2}\cdot\frac{dz}{dq_r}\right),$$

$$\Sigma m\left(\frac{dx}{dt}\cdot\frac{dx}{dq_r}+\frac{dy}{dt}\cdot\frac{dy}{dq_r}+\frac{dz}{dt}\cdot\frac{dz}{dq_r}\right),$$

and the action $$\int\Sigma.m\left(\frac{dx}{dt}\,dx+\frac{dy}{dt}\,dy+\frac{dz}{dt}\,dz\right),$$

are also invariants.

Again, in the case where a, b, c, &c., are replaced by P, Q, R, U, V, W, the components of stress, the quadric $U=c$ is the stress quadric, and $U'=0$ the cone of shearing stress. The invariants are

$$P+Q+R=D,$$

$$PQ+QR+RP-U^2-V^2-W^2=S,$$

$$PQR+2UVW-PU^2-QV^2-RW^2=\Delta.$$

If the system be equivalent to a shearing stress, D and Δ vanish, and $-S$ is the square of the shearing stress.

D and S are the S and Σ in Clapeyron's Theorem (Minchin's *Statics*, p. 515), and the expression

$$\frac{1}{2\mu}\int(P^2+Q^2+R^2+2U^2+2V^2+2W^2)\,dw,$$

is equivalent to $$\frac{1}{2\mu}\int(D^2-2S)\,dw.$$

The consideration of another important class of functions, which includes the components of strain, will come more conveniently after that of the combinants of two sets of functions.

VI. When there are two sets of quantities a, b, c, &c., and a', b', c', &c., both transforming as before, the invariants D^2-2S and Δ give rise to three combinants

$$aa'+bb'+cc'+2ff'+2gg'+2hh'\equiv E\text{ say},$$

$$a'bc+ab'c+abc'+2f'gh+2fg'h+2fgh'$$
$$-a'f^2-b'g^2-c'h^2-2aff'-2bgg'-2chh'\equiv\Theta,$$

and $$ab'c'+a'bc'+a'b'c+2fg'h'+2f'gh'+2f'g'h$$
$$-af'^2-bg'^2-ch'^2-2a'f'f-2b'g'g-2c'h'h=\Theta'.$$

If a', b', &c., be proportional to u^2, v^2, &c., as in (III.),

$$E=au^2+bv^2+cw^2+2fvw+2gwu+2huv,$$

a form which shows that the result of putting any quantities cogredient with x, y, z for these in U, is unchanged by transformation. The same substitution gives

$$\Theta=Au^2+Bv^2+Cw^2+2Fvw+2Gwu+2Huv,$$

where A, B, C, &c. have the same meaning as in (I.) So that θ is the result of writing u, v, w for x, y, z in U'. With these values of a', b', &c., Θ' vanishes identically, or, if a', b', &c. be proportional to uu', vv', &c., as in (IV.),

$$E=auu'+bvv'+cww'+f(vw'+v'w)+g(wu'+w'u)+h(uv'+u'v)$$
$$=(au'+hv'+fw')u+(hu'+bv'+fw')v+(gu'+fv'+cw')w,$$

and $$\Theta=(Au'+Hv'+Fw')u+(Hu'+Bv'+Fw')v+(Gu'+Fv'+Cw')w,$$

and $$-4\Theta'=a(vw'-v'w)^2+b(wu'-w'u)^2+c(uv'-u'v)^2$$
$$+2f(wu'-w'u)(uv'-u'v)+2g(uv'-u'v)(vw'-v'w)$$
$$+2h(vw'-v'w)(wu'-w'u),$$

the result of writing $vw'-v'w$, &c. for x, y, z in U. So that this is, in fact, a case of the value of E obtained before. Examples of most of these combinants occur in the theory of Quadrics.

Again, if a, b, c, &c. be the components of strain, a', b', c', &c., those of stress, $E\,dw$ is the element of the work done in strain. With the same suppositions, the expression

$$a'da+b'db+c'dc+2f'df+2g'dg+2h'dh,*$$

a quantity employed by Sir W. Thomson, and which he has shown must be an exact differential, is a form of the combinant E. The two quantities $A\omega_1^2+B\omega_2^2+C\omega_3^2$ and $A^2\omega_1^2+B^2\omega_2+C^2\omega_3^2$, A, B, and C being the principal moments of inertia, which occur in the solution of Euler's equations of motion in the case where there are no impressed forces, are both instances of E. The quantities a, b, c, &c., and a', b', c', &c., being, for the first, A, B, C, $-F$, $-G$, $-H$, when these expressions represent the moments and products of inertia, and ω_1^2, ω_2^2, ω_3^2, $\omega_2\omega_3$, &c., where ω_1, ω_2, ω_3 are the angular velocities about the axes. So that

$$E=A\omega_1^2+B\omega_2^2+C\omega_3^2-2F\omega_2\omega_3-2G\omega_3\omega_1-2H\omega_1\omega_2,$$

which reduces to $$A\omega_1^2+B\omega_2^2+C\omega_3^2,$$

* I am indebted for my acquaintance with this function to a remark made by Professor C. Niven.

when the axes are principal. For the second, they are ω_1^2, &c., as before, and the coefficients of a covariant U''' of the momental ellipsoid U, which may be obtained as follows.

Let U' denote the ellipsoid of gyration

$$(BC-F^2)\,x^2+(CA-G^2)\,y^2+(AB-H^2)\,z^2-2\,(GH+AF)\,yz$$
$$-2\,(HF+BG)\,zx-2\,(FG+CH)\,xy.$$

Then
$$\begin{aligned}U'''&=(A+B+C)\,U+U'\\&\quad-(AB+BC+CA-F^2-G^2-H^2)\,(x^2+y^2+z^2)\\&\equiv(A^2+G^2+H^2)\,x^2+(B^2+F^2+H^2)\,y^2+(C^2+G^2+F^2)\,z^2\\&\quad-2\,(2AF+BF+CF+GH)\,yz\\&\quad-2\,(2BG+CG+AG+HF)\,zx\\&\quad-2\,(2CH+AH+BH+FG)\,xy;\end{aligned}$$

so that
$$E=(A^2+G^2+H^2)\,\omega_1^2+\&c.,$$
$$-2\,(2AF+BF+CF+GH)\,\omega_2\,\omega_3-\&c.,$$

which reduces to $A^2\omega_1^2+B^2\omega_2^2+C^2\omega_3^2$ when the axes are principal. The equations $U'''=\lambda^6$ and $U=\epsilon^4$ determine the polhode. The next section will furnish some further examples of the combinants.

VII. The special class already mentioned is that in which a, b, c, f, g, h, are proportional to $\frac{du}{dx}$, $\frac{dv}{dy}$, $\frac{dw}{dz}$, $\frac{1}{2}\left(\frac{dv}{dz}+\frac{dw}{dy}\right)$, $\frac{1}{2}\left(\frac{dw}{dx}+\frac{du}{dz}\right)$, $\frac{1}{2}\left(\frac{du}{dy}+\frac{dv}{dx}\right)$ respectively, u, v, w being quantities cogredient with x, y, and z. In this case it appears from the number of equations that there ought to be three additional invariants. These may be derived as follows. Since $\frac{dw}{dy}-\frac{dv}{dz}$, $\frac{du}{dz}-\frac{dw}{dx}$, $\frac{dv}{dx}-\frac{du}{dy}$ are cogredient with x, y, z,

$$\left(\frac{dw}{dy}-\frac{dv}{dz}\right)^2+\left(\frac{du}{dz}-\frac{dw}{dx}\right)^2+\left(\frac{dv}{dx}-\frac{du}{dy}\right)^2=\Omega^2 \text{ say,}$$

is an invariant.

This combined with S gives

$$\frac{du}{dx}\cdot\frac{dv}{dy}+\frac{dv}{dy}\cdot\frac{dw}{dz}+\frac{dw}{dz}\cdot\frac{du}{dx}$$
$$-\frac{du}{dy}\cdot\frac{dv}{dx}-\frac{dv}{dz}\cdot\frac{dw}{dy}-\frac{dw}{dx}\cdot\frac{du}{dz}=\frac{4S+\Omega^2}{4}.$$

Then the combinant E of

$$\left(\frac{du}{dx},\ \frac{dv}{dy},\ \ldots\ldots\right)(x, y, z)^2$$

and
$$\left\{\left(\frac{dw}{dy}-\frac{dv}{dz},\ \frac{du}{dz}-\frac{dw}{dx},\ \frac{dv}{dx}-\frac{du}{dy}\right)(x, y, z)\right\}^2,$$

or $$\frac{du}{dx}\left(\frac{dw}{dy}-\frac{dv}{dz}\right)^2+\frac{dv}{dy}\left(\frac{du}{d}-\frac{dw}{dx}\right)^2+\frac{dw}{dz}\left(\frac{dv}{dx}-\frac{du}{dy}\right)^2$$
$$+\left(\frac{dw}{dy}+\frac{dv}{dz}\right)\left(\frac{du}{dz}-\frac{dw}{dx}\right)\left(\frac{dv}{dx}-\frac{du}{dy}\right)$$
$$+\left(\frac{du}{dz}+\frac{dw}{dx}\right)\left(\frac{dv}{dx}-\frac{du}{dy}\right)\left(\frac{dw}{dy}-\frac{dv}{dz}\right)$$
$$+\left(\frac{dv}{dx}+\frac{du}{dy}\right)\left(\frac{dw}{dy}-\frac{dv}{dz}\right)\left(\frac{du}{dz}-\frac{dw}{dx}\right)=H \text{ say,}$$

which combined with Δ gives

$$\begin{vmatrix} \frac{du}{dx} & \frac{du}{dy} & \frac{du}{dz} \\ \frac{dv}{dx} & \frac{dv}{dy} & \frac{dv}{dz} \\ \frac{dw}{dx} & \frac{dw}{dy} & \frac{dw}{dz} \end{vmatrix} = \Delta+\frac{H}{4},$$

the Jacobian of u, v, and w.

The third additional invariant is the combinant Θ of the same functions

$$\left(\frac{dw}{dy}-\frac{dv}{dz}\right)^2\frac{dv}{dy}\cdot\frac{dw}{dz}+\left(\frac{du}{dz}-\frac{dw}{dx}\right)^2\frac{dw}{dz}\cdot\frac{du}{dx}$$
$$+\left(\frac{dv}{dx}-\frac{du}{dy}\right)^2\frac{du}{dx}\cdot\frac{dv}{dy}-\frac{du}{dx}\left(\frac{dw}{dy}+\frac{dv}{dz}\right)\left(\frac{du}{dz}-\frac{dw}{dx}\right)\left(\frac{dv}{dx}-\frac{du}{dy}\right)$$
$$-\frac{dv}{dy}\left(\frac{du}{dz}+\frac{dw}{dx}\right)\left(\frac{dv}{dx}-\frac{du}{dy}\right)\left(\frac{dw}{dy}-\frac{dv}{dz}\right)$$
$$-\frac{dw}{dz}\left(\frac{dv}{dx}+\frac{du}{dy}\right)\left(\frac{dw}{dy}-\frac{dv}{dz}\right)\left(\frac{du}{dz}-\frac{dw}{dx}\right)$$
$$-\frac{1}{4}\left\{\left(\frac{dw^2}{dy^2}-\frac{dv^2}{dz^2}\right)^2+\left(\frac{du^2}{dz^2}-\frac{dw^2}{dx^2}\right)^2+\left(\frac{dv^2}{dx^2}-\frac{du^2}{dy^2}\right)^2\right.$$
$$-2\left(\frac{dw^2}{dy^2}-\frac{dv^2}{dz^2}\right)\left(\frac{du^2}{dz^2}-\frac{dw^2}{dx^2}\right)-2\left(\frac{du^2}{dz^2}-\frac{dw^2}{dx^2}\right)\left(\frac{dv^2}{dx^2}-\frac{du^2}{dy^2}\right)$$
$$\left.-2\left(\frac{dv^2}{dx^2}-\frac{du^2}{dy^2}\right)\left(\frac{dw^2}{dy^2}-\frac{dv^2}{dz^2}\right)\right\}=Z \text{ say.}$$

H and Z both vanish if Ω does.

Whenever two vectors whose components are u, v, w and x, y, z, are so related that one of them is a linear and vector function of the other, the nine quantities $\frac{du}{dx}$, $\frac{du}{dy}$, &c. are the coefficients in the expressions for u, v, w in terms of x, y, and z, and the six a, b, c, &c. (with the values given above) are those in the value of the invariant $ux+vy+wz$, in terms of x, y, and z. The strains of a solid or fluid body are one

important set of functions of this kind. In this case

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=D, \text{ the dilatation,}$$

is the coefficient of ρ in the differential equation of continuity, $\Delta+\frac{H}{4}$ being that in the integral equation. Ω is the resultant rotation, and D, $S+\frac{\Omega^2}{4}$, $\Delta+\frac{H}{4}$ are the coefficients in the equation which determines the lines of no rotation. $\Omega=0$ is the condition for steady motion, *i.e.*, the condition that the invariant

$$u\,dx+v\,dy+w\,dz$$

should be an exact differential. This involves the vanishing of H and Z.

In the case of a strain, $\Omega=0$ is the condition that it should be pure.

$D=0$ and $\Delta=0$ that it should be equivalent to a shear, and in this case $-S$ is the square of the shear.

$\Delta=0$ that there should be two planes of no elongation. And this, together with $S=0$, that the strain should be equivalent to a simple elongation.

When
$$u=\frac{dF}{dx},\quad v=\frac{dF}{dy}\quad w=\frac{dF}{dz},$$

$$D=\frac{d^2F}{dx^2}+\frac{d^2F}{dy^2}+\frac{d^2F}{dz^2},$$

the important quantity on the left-hand side of Lagrange's and Poisson's equations.

$$S=\frac{d^2F}{dx^2}\cdot\frac{d^2F}{dy^2}+\frac{d^2F}{dy^2}\cdot\frac{d^2F}{dz^2}+\frac{d^2F}{dz^2}\cdot\frac{d^2F}{dx^2}$$
$$-\left(\frac{d^2F}{dx\,dy}\right)^2-\left(\frac{d^2F}{dy\,dz}\right)^2-\left(\frac{d^2F}{dz\,dx}\right)^2,$$

and Δ is the Hessian of F.

The class of linear and vector functions of vectors includes several important quantities which appear in the theory of Electricity and Magnetism—as the Electromotive Intensity which is a function of the Electric displacement (Clerk Maxwell's *Electricity and Magnetism*, Vol. i., p. 136); the Electromotive Force of the Current (p. 383); the Magnetisation of the Magnetic Force (Vol. ii., p. 60). In every case the six invariant relations hold among the coefficients.

If
$$u=u_1x+u_2y+u_3z,\quad v=v_1x+v_2y+v_3z,$$
$$w=w_1x+w_2y+w_3z,$$

be the equations connecting the vectors, when referred to any rectangular axes, the coefficients corresponding to any others with same

origin may be obtained by transforming

$$(u_1x+u_2y+u_3z)\,u+(v_1x+v_2y+v_3z)\,v+(w_1x+u_2y+u_3z)\,w,$$

u, v, w and x, y, z being transformed by the same transformation. In all cases, the coefficients of such a function will have the six invariants given above.

VIII. The invariants may also be made use of in other ways. Thus—

(*a.*) To transform the equation $U=\mu$ to the form

$$H(x^2+y^2+z^2)+\lambda x^2+2kyz=\mu$$

(a form suggested by Thomson and Tait as a canonical one for strains, &c.)

Here $$3H+\lambda=D,\quad 3H^2+2H\lambda-k^2=S,$$

and $$H^3+H^2\lambda-(H+\lambda)\,k^2=\Delta.$$

Hence $$(H+\lambda)^3-D(H+\lambda)^2+S(H+\lambda)-\Delta=0.$$

So that $H+\lambda$ is a root of the discriminating cubic; and, therefore equal to one of the principal coefficients (the squares of the reciprocals of the principal axes). Let these be A, B, and C.

$$\text{If } H+\lambda=A,\ H=\frac{B+C}{2},\ \lambda=A-\frac{B+C}{2},\ \text{and } k^2=\left(\frac{B-C}{2}\right)^2.$$

If λ is to be positive, A must be greater than $\frac{B+C}{2}$ which may be secured by supposing $\frac{1}{\sqrt{A}}$ the least axis.

Comparing the resulting form of the equation

$$Ax'^2+\frac{B+C}{2}(y'^2+z'^2)\pm(B-C)\,y'z'=\mu$$

with $Ax^2+By^2+Cz^2=\mu$, it is evident that the axis of x is unchanged; and if $y'=\lambda y+\mu z$, $z'=\mu y-\lambda z$ be the equations between the new and old coordinates, $\lambda=\mu=\frac{1}{\sqrt{2}}$. So that the new axes of y and z bisect the angles between the old.

(*b.*) If $U=\mu$ be a spheroid, the discriminating cubic must have equal roots, and therefore

$$(DS-9\Delta)^2=4(D^2-3S)(S^2-3D\Delta).$$

If U can be reduced to $x^2+\lambda(y^2+z^2)$, $D=1+2\lambda$, $S=2\lambda+\lambda^2$, $\Delta=\lambda^2$, and therefore

$$4(S+1)=(D+1)^2=4(D+\Delta),\ \text{and } \lambda=\frac{D-1}{2}.$$

(*c.*) The conditions that two quadrics, the coefficients in the equations of which are a, b, c, &c., and a', b', c', &c., should be similar, may be obtained from the consideration that in this case the roots of $t^3-Dt^2+St-\Delta=0$ must each be the same multiple of the corresponding root of $t^3-D't^2+S't-\Delta'=0$, and they are

$$\frac{D}{D}=\sqrt{\frac{S}{S'}}=\sqrt[3]{\frac{\Delta}{\Delta'}}.$$

(*d.*) It is shown in Salmon's *Geometry of Three Dimensions*, p. 155, that $D=0$ is the condition that it should be possible to draw three rectangular asymptotic lines to the surface; and $S=0$, that it should be possible to draw three rectangular asymptotic planes. This follows from the previous statement, since S is the D of U', the polar reciprocal. $\Delta=0$ is, as was pointed out in (IV.), the condition that $U=\mu$ should be a cylinder, and, since the roots of the discriminating cubic are all real,

$$4(D^2-3S)(S^2-3D\Delta)-(DS-9\Delta)^2,$$

its discriminant with the sign changed, is positive.

IX. The invariants may also be used to show that every motion of a rigid body of which one point is fixed, is a rotation about an axis; and that the most general motion of a rigid body may be reduced to a twist about a screw. (Ball's *Theory of Screws*, p. 3.)

If a rigid body have one point fixed, and x, y, z; ξ, η, ζ be the old and new coordinates of any other point in it referred to the same rectangular axes, with the fixed point as origin. Then

$$\xi=l_1x+m_1y+n_1z,\quad \eta=l_2x+m_2y+n_2z,$$

$$\zeta=l_3x+m_3y+n_3z,$$

l_1, l_2, l_3; m_1, m_2, m_3; n_1, n_2, n_3 being the direction cosines of the three lines (fixed in the body), which originally coincided with the axes. Then $x\xi+y\eta+z\zeta$ is not altered by transformation to other axes with the same origin; and its equivalent

$$l_1x^2+m_2y^2+n_3z^2+(n_2+m_3)\,yz+(l_3+n_1)\,zx+(m_1+l_2)\,xy$$

is also unaltered. It is easy to show that the invariants of this expression satisfy the conditions (VIII. *b*)

$$4(S+1)=(D+1)^2=4(\Delta+D),$$

and therefore the expression can be reduced, by a proper choice of axes, to the form

$$x'^2+\lambda(y'^2+z'^2),\text{ where }\lambda=\frac{D-1}{2}.$$

Therefore, if x', y', z' and ξ', η', ζ' be the old and new coordinates of the point whose original coordinates were x, y, z.

$$\xi' = x', \quad \eta' = \lambda y' - \mu z', \quad \zeta' = \mu y' + \lambda z',$$

where $\lambda^2 + \mu^2 = 1$. So that the motion is reduced to a rotation about the axis of x', through an angle

$$= \cos^{-1}\left(\frac{D-1}{2}\right).$$

If $x' = a_1 x + a_2 y + a_3 z$, and therefore $\xi' = a_1 \xi + a_2 \eta + a_3 \zeta$, so that a_1, a_2, a_3 are the direction cosines of the axis of rotation; the equation $x' = \xi'$ leads to

$$(l_1 - 1)\, a_1 + l_2 a_2 + l_3 a_3 = 0,$$
$$m_1 a_1 + (m_2 - 1)\, a_2 + m_3 a_3 = 0,$$
$$n_1 a_1 + n_2 a_2 + (n_3 - 1)\, a_3 = 0,$$

three equations which are consistent with each other, and which determine $a_1 : a_2 : a_3$; and the equations to the axis may be put in the form

$$(l_1 - 1)\, x + l_2 y + l_3 z = 0,$$
$$m_1 x + (m_2 - 1)\, y + m_3 z = 0,$$
$$n_1 x + n_2 y + (n_3 - 1)\, z = 0,$$

any two of which determine it, or

$$(n_2 + m_3)\, x = (l_3 + n_1)\, y = (m_1 + l_2)\, z.$$

Again, if x, y, z and ξ, η, ζ be the original and final coordinates of any point in a rigid body referred to the same fixed axes; the most general motion may be denoted by

$$\xi = u + l_1 x + m_1 y + n_1 z, \quad \eta = v + l_2 x + m_2 y + n_2 z,$$
$$\zeta = w + l_3 x + m_3 y + n_3 z,$$

l_1, &c. having the same signification as before, and u, v, w being the displacements of the point originally at the origin parallel to the axes. Then the invariable quantity

$$x\,(\xi - u) + y\,(\eta - v) + z\,(\zeta - w)$$

may be reduced as above; so that, x', y', z' and ξ', η', ζ' being the original and final coordinates referred to the new axes, which are determined exactly as before,

$$\xi' = u' + x', \quad \eta' = v' + \lambda y' - \mu z', \quad \zeta' = w' + \mu y' + \lambda z';$$

and, by changing the origin to $(0, k, l)$, if x'', y'', z'' and ξ'', η'', ζ'' be the new coordinates,

$$\xi'' = u' + x'',$$
$$\eta'' = v' + \lambda\,(y'' + k) - \mu\,(z'' + l) = \lambda y'' - \mu z'',$$
$$\zeta'' = w' + \mu\,(y'' + k) + \lambda\,(z'' + l) = \mu y'' + \lambda z''.$$

If k and l be determined by the equations

$$\left.\begin{aligned}\lambda k-\mu l+v'&=0\\ \mu k+\lambda l+w'&=0\end{aligned}\right\} \text{ or } \left\{\begin{aligned}k&=-(\lambda v'+\mu w'),\\ l&=\ \ \mu v'-\lambda u',\end{aligned}\right.$$

this proves that the motion can always be reduced to a twist about a screw, whose pitch

$$=\frac{u'}{\cos^{-1}\left(\frac{D-1}{2}\right)}, \text{ where } u'=\frac{1}{\sqrt{-S}}\{(n_2+m_3)u+(l_3+n_1)v+(m_1+l_2)w\},$$

and the axis of which is parallel to the line

$$(n_2+m_3)x=(l_3+n_1)y=(m_1+l_2)z$$

through the point (x', y', z'), where

$$x'=0,\quad y'=k,\quad z'=l.$$

X. Whatever may be the correct expression of stress in terms of strain, the invariants of stress and the combinants of strain and stress will be expressible in terms of the invariants of strain.

If, then, it be assumed that the stresses are linear functions of the strains (*i.e.*, that these last are small), these conditions may be applied to find relations among the coefficients.

Let $\overline{P}$, $\overline{Q}$, $\overline{R}$, $\overline{U}$, $\overline{V}$, $\overline{W}$ denote the stresses (chief stresses, say) which correspond to the principal strains A, B, C, and let

$$\begin{aligned}\overline{P}&=p_1A+p_2B+p_3C,\\ \overline{Q}&=q_1A+q_2B+q_3C,\\ \overline{R}&=r_1A+r_2B+r_3C,\\ \overline{U}&=u_1A+u_2B+u_3C,\\ \overline{V}&=v_1A+v_2B+v_3C,\\ \overline{W}&=w_1A+w_2B+w_3C.*\end{aligned}$$

Then, denoting the invariants of stress by D', S', and Δ', those of strain by D, S, and Δ, and the combinants by E, Θ, and Θ',—

$$\begin{aligned}D'&=\alpha D \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1),\\ S'&=\beta D^2+\gamma S \qquad\qquad\qquad\qquad\qquad\qquad (2),\\ \Delta'&=\delta D^3+\epsilon DS+\zeta\Delta \qquad\qquad\qquad\qquad (3),\\ E&=\beta' D^2+\gamma' S \qquad\qquad\qquad\qquad\qquad (4),\\ \Theta&=\delta' D^3+\epsilon' DS+\zeta'\Delta \qquad\qquad\qquad (5),\\ \Theta'&=\delta'' D^3+\epsilon'' DS+\zeta''\Delta \qquad\qquad\qquad (6),\end{aligned}$$

* The eighteen coefficients p_1, p_2, p_3, q_1, &c., together with three additional quantities to determine the direction of the principal axes, make up twenty-one quantities upon which the general values of the coefficients in the expressions for stress in terms of strain depend.

a series of equations which give

$$p_1+q_1+r_1 = p_2+q_2+r_2 = p_3+q_3+r_3, \text{ from (1)},$$

$$p_1 = q_2 = r_3,\ p_2+q_1 = p_3+r_1 = q_3+r_2, \text{ from (4)},$$

and from (5),

$$*p_2 = p_3 = q_1 = q_3 = r_1 = r_2,$$

a set of values which will satisfy all the equations when $\overline{U} = 0$, $\overline{V} = 0$, $\overline{W} = 0$, that is, when the principal axes of strain coincide with those of stress. In this case, if $p_1 = \&c. = k$, $p_2 = p_3 = \&c. = \mu$,

$$\overline{P} = kA+\mu(B+C),$$
$$\overline{Q} = kB+\mu(C+A),$$
$$\overline{R} = kC+\mu(A+B);$$

and hence, by transformation to any axes,

$$P = ka+\mu(b+c),\quad Q = kb+\mu(c+a),\quad R = kc+\mu(a+b),$$
$$U = (k-\mu)f,\quad V = (k-\mu)g,\quad W = (k-\mu)h;$$

in which expressions for the stresses the coefficients are independent of the direction of the axes. It follows that, in this case, the body is isotropic.† The mechanical meaning of the constants may be deduced from the following considerations :—

(i.) If the strain be such that $a+b+c = 0$, there is no expansion, and the strain is wholly distortion. In this case,

$$P = (k-\mu)a,\quad Q = (k-\mu)b,\quad R = (k-\mu)c, \&c.,$$

and
$$E = (k-\mu)[a^2+b^2+c^2+2f^2+2g^2+2h^2]$$
$$= (k-\mu)[D^2-2S],$$

and $k-\mu$ must be the resistance to distortion.

(ii.) If the strain be a uniform dilatation, there will be no distortion. In this case $E = 3(k+2\mu)a^2 = (k+2\mu)S$, and $k+2\mu$ is the resistance to elongation,—which are in fact the known results in the case of isotropic bodies.

Returning to the general case, when the principal axes of strain and stress do not coincide, if, as before, k be put for p_1, q_2, or r_3, and μ for p_2, p_3,

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* Hence $\overline{P}dA + \overline{Q}dB + \&c.$, is a perfect differential, and as this expression is an invariant, $Pda + Qdb + \&c.$, is always so, as remarked by Sir W. Thomson.

† This is the converse of Cauchy's assumption that the axes of strain and stress coincide in an isotropic body.

q_1, q_2, r_1, or r_2;

$$\begin{aligned}
S' &= \overline{P}\cdot\overline{Q} + \overline{Q}\cdot\overline{R} + \overline{R}\cdot\overline{P} - \overline{U}^2 - \overline{V}^2 - \overline{W}^2 \\
&= (2k\mu+\mu^2-u_1^2-v_1^2-w_1^2)\,A^2 \\
&+(2k\mu+\mu^2-u_2^2-v_2^2-w_2^2)\,B^2+(2k\mu+\mu^2-u_3^2-v_3^2-w_3^2)\,C^2 \\
&+(k^2+2k\mu+3\mu^2-2u_1u_2-2v_1v_2-2w_1w_2)\,A\,.\,B \\
&+(k^2+2k\mu+3\mu^2-2u_2u_3-2v_2v_3-2w_2w_3)\,B\,.\,C \\
&+(k^2+2k\mu+3\mu^2-2u_3u_1-2v_3v_1-2w_3w_1)\,C\,.\,A,
\end{aligned}$$

which, by (2), leads to

$$\begin{aligned}
&u_1^2+v_1^2+w_1^2 = u_2^2+v_2^2+w_2^2 = u_3^2+v_3+w_3^2, \\
&u_1u_2+v_1v_2+w_1w_2 = u_2u_3+v_2v_3+w_2w_3 = u_3u_1+v_3v_1+w_3w_1, \\
&\Theta' = A\overline{Q}\overline{R}+B\overline{R}\overline{P}+C\overline{P}\overline{Q}-A\overline{U}^2-B\overline{V}^2-C\overline{W}^2 \\
&\quad = A^3(\mu^2-u_1^2)+B^3(\mu^2-v_2^2)+C\ (\mu^2-w_3^2) \\
&\qquad +A^2B\,(\mu^2+2k\mu-2u_1u_2-v_1^2)+AB^2\,(\mu^2+2k\mu-2v_1v_2-u_2^2) \\
&\qquad +A^2C\,(\mu^2+2k\mu-2u_1u_3-w_1^2)+\&c. \\
&\qquad +ABC\,(3k^2+3\mu^2-2u_2u_3-2v_3v_1-2w_1w_2),
\end{aligned}$$

and therefore, by (5),

$$\begin{aligned}
&u_1^2 = v_2^2 = w_3^2 \quad\text{and}\quad 2u_1u_2+v_1^2 = 2v_1v_2+u_2^2 \\
&\quad = 2u_1u_3+w_1^2 = 2w_1w_3+u_3^2 = 2v_2v_3+w_2^2 = 2w_2w_3+v_3^2,
\end{aligned}$$

and

$$\begin{aligned}
\Delta' &= \overline{P}\overline{Q}\overline{R}+2\overline{U}\overline{V}\overline{W}-\overline{P}\overline{U}^2-\overline{Q}\overline{V}^2-\overline{R}\overline{W}^2 \\
&= A^3\{k\mu^2+2u_1v_1w_1-ku_1^2-\mu v_1^2-\mu w_1^2\} \\
&\quad +B^3\{k\mu^2+2u_2v_2w_2-kv_2^2-\mu w_2^2-\mu u_2^2\} \\
&\quad +C^3\{k\mu^2+2u_3v_3w_3-kw_3^2-\mu u_3^2-\mu v_3^2\} \\
&+A^2B\{k^2\mu+k\mu^2+\mu^3+2u_1v_1w_2+2u_2v_1w_1+2u_1v_2w_1 \\
&\qquad -2ku_1u_2-\mu u_1^2-kv_1^2-2\mu v_1v_2-\mu w_1^2-2\mu w_1w_2\} \\
&+AB^2\{k^2\mu+k\mu^2+\mu^3+2u_2v_2w_1+2u_1v_2w_2+2u_2v_1w_2 \\
&\qquad -2kv_1v_2-\mu v_2^2-ku_2^2-2\mu u_1u_2-\mu w_2^2-2\mu w_1w_2\} \\
&+A^2C\{k^2\mu+k\mu^2+\mu^3+2u_1v_1w_3+2u_3v_1w_1+2u_1v_3w_1 \\
&\qquad -2ku_1u_3-\mu u_1^2-kw_1^2-2\mu w_1w_3-\mu v_1^2-2\mu v_1v_3\} \\
&\qquad +\&c. \\
&+ABC\{k^3+3k\mu^2+2\mu^3+2u_1v_2w_3+2u_2v_1w_3+2u_3v_1w_2 \\
&\qquad +2u_1v_3w_2+2u_3v_2w_1+2u_2v_3w_1-2ku_2u_3-2\mu u_1u_2-2\mu u_3u_1 \\
&\qquad -2kv_3v_1-2\mu v_1v_2-2\mu v_2v_3-2kw_1w_2-2\mu w_3w_1-2\mu w_3w_2\},
\end{aligned}$$

which, by (3), using the previous equations among the u's, v's, and

w's, shows that

$$u_2v_1w_1+u_1v_2w_1+u_1v_1w_2 = u_3v_1w_1+u_1v_3w_1+u_1v_1w_3$$
$$= u_1v_2w_3+u_2v_1w_2+u_2v_2w_1 = \&c.,$$

and
$$u_1v_1w_1 = u_2v_2w_2 = u_3v_3w_3,$$

and therefore
$$(u_1-u_2)(v_1-v_2)(w_1-w_2) = 0,$$
$$(u_1-u_3)(v_1-v_3)(w_1-w_3) = 0,$$
$$(u_2-u_3)(v_2-v_3)(w_2-w_3) = 0,$$

and ultimately
$$u_1^2 = v_2^2 = w_3^2 = \lambda^2 \text{ suppose,}$$
$$u_2^2 = u_3^2 = v_1^2 = v_3^2 = w_1^2 = w_3^2 = \nu^2,$$

all the u's, all the v's, and all the w's being respectively of the same sign.

Assuming that they all have the positive sign, the values of the chief strains are

$$\overline{P} = kA+\mu(B+C),$$
$$\overline{Q} = kB+\mu(C+A),$$
$$\overline{R} = kC+\mu(A+B),$$
$$\overline{U} = \lambda A+\nu(B+C),$$
$$\overline{V} = \lambda B+\nu(C+A),$$
$$\overline{W} = \lambda C+\nu(A+B),$$

and those of the invariants and combinants are

$$D' = (k+2\mu)D,$$
$$S' = (2k\mu+\mu^2-\lambda^2-\nu^2)D^2 + \{(k-\mu)^2+2(\lambda-\nu)^2\}S,$$
$$\Delta' = (k\mu^2-k\lambda^2+2\lambda\nu^2-2\mu\nu^2)D^3 + (k^2\mu-2k\mu^2+\mu^3+3k\lambda^2-2k\lambda\nu-k\nu^2+2\lambda^2\nu - 4\lambda\nu^2+2\nu^3-2\mu\lambda^2-2\mu\lambda\nu+4\mu\nu^2)DS + (k^3-3k^2\mu+3k\mu^2-\mu^3-3k\lambda^2+6k\lambda\nu-3k\nu^2 + 2\lambda^3-6\lambda^2\nu+6\lambda\nu^2-2\nu^3+6\mu\lambda^2-6\mu\lambda\nu)\Delta,$$

$$E = kD^2-2(k-\mu)S,$$
$$\Theta = 3(k-\mu)\Delta+\mu DS,$$
$$\Theta' = (\mu^2-\lambda^2)D^3+(2k\mu-2\mu^2+3\lambda^2-2\lambda\nu-\nu^2)DS + 3\{(k-\mu)^2-(\lambda-\nu)^2\}\Delta.$$

XI. The same method is no longer practically available, when the axes are not principal. For, since the quadric of which the coefficients are a, b, &c., has principal axes, there will be a set of quantities of the

nature of covariants; that is, if $l_1 m_1 n_1$, $l_2 m_2 n_2$, $l_3 m_3 n_3$ be the direction cosines of the principal axes,

$$\begin{aligned} F = 0 = {} & a l_2 l_3 + b m_2 m_3 + c n_2 n_3 + f (m_2 n_3 + m_3 n_2) \\ & + g (n_2 l_3 + n_3 l_2) + h (l_2 m_3 + l_3 m_2), \\ G = 0 = {} & a l_3 l_1 + b m_3 m_1 + c n_3 n_1 + f (m_1 n_3 + m_3 n_1) \\ & + g (n_3 l_1 + n_1 l_3) + h (l_3 m_1 + l_1 m_3), \\ H = 0 = {} & a l_1 l_2 + b m_1 m_2 + c n_1 n_2 + f (m_1 n_2 + m_2 n_1) \\ & + g (n_1 l_2 + n_2 l_1) + h (l_1 m_2 + l_2 m_1), \end{aligned}$$

as well as the expressions corresponding to these and derived from the covariant quadrics, which are, however, evidently not independent relations like those above. These all vanished identically when the axes were principal, and did not require to be noticed, but in any other case functions of the values of F, G, and H, must be added to the expressions for the components of stress. This would introduce an unmanageable number of terms into the equations, which, without altering the expressions for the invariants and combinants of stress, would affect the coefficients of the various strain components in the values of those of stress.

The coefficients for any axes may, however, be determined as follows, in terms of k, μ, λ, and ν, the constants already employed, and the direction cosines of the principal axes of strain with respect to those to which the strain is actually referred, and which involve three additional constants. Let these be $(l_1 m_1 n_1)$, &c., so that $(l_1 l_2 l_3)$, $(m_1 m_2 m_3)$, $(n_1 n_2 n_3)$ are the direction cosines of the axes referred to the principal axes. Then

$$\begin{aligned} P = {} & \overline{P} l_1^2 + \overline{Q} l_2^2 + \overline{R} l_3^2 + \overline{U} l_2 l_3 + 2 \overline{V} l_3 l_1 + 2 \overline{W} l_1 l_2 \\ = {} & A (k l_1^2 + \mu l_2^2 + \mu l_3^2 + 2\lambda l_2 l_3 + 2\nu l_3 l_1 + 2\nu l_1 l_2) \\ & + B (\mu l_1^2 + k l_2^2 + \mu l_3^2 + 2\nu l_2 l_3 + 2\lambda l_3 l_1 + 2\nu l_1 l_2) \\ & + C (\mu l_1^2 + \mu l_2^2 + k l_3^2 + 2\nu l_2 l_3 + 2\nu l_3 l_1 + 2\lambda l_1 l_2) \\ \equiv {} & P_1 A + P_2 B + P_3 C \text{ suppose} \\ = {} & P_1 (a l_1^2 + b m_1^2 + c n_1^2 + 2f m_1 n_1 + 2g n_1 l_1 + 2h l_1 m_1) \\ & + P_2 (a l_2^2 + b m_2^2 + c n_2^2 + 2f m_2 n_2 + 2g n_2 l_2 + 2h l_2 m_2) \\ & + P_3 (a l_3^2 + b m_3^2 + c n_3^2 + 2f m_3 n_3 + 2g n_3 l_3 + 2h l_3 m_3) \\ = {} & a (P_1 l_1^2 + P_2 l_2^2 + P_3 l_3^2) + b (P_1 m_1^2 + P_2 m_2^2 + P_3 m_3^2) \\ & + c (P_1 n_1^2 + P_2 n_2^2 + P_3 n_2^2) + 2f (P_1 m_1 n_1 + P_2 m_2 n_2 + P_3 m_3 n_3) \\ & + 2g (P_1 n_1 l_1 + P_2 n_2 l_2 + P_3 n_3 l_3) + 2h (P_1 l_1 m_1 + P_2 l_2 m_2 + P_3 l_3 m_3), \end{aligned}$$

with similar expressions for Q and R.

$$\begin{aligned}U = {} & \overline{P} m_1 n_1 + \overline{Q} m_2 n_2 + \overline{R} m_3 n_3 + \overline{U}\,(m_3 n_2 + m_2 n_3) \\ & + \overline{V}\,(m_3 n_1 + m_1 n_3) + \overline{W}\,(m_1 n_2 + m_2 n_1) \\ = {} & A\{k m_1 n_1 + \mu m_2 n_2 + \mu m_3 n_3 + \lambda\,(m_3 n_2 + m_2 n_3) \\ & + \nu\,(m_3 n_1 + m_1 n_3) + \nu\,(m_1 n_2 + m_2 n_1)\} \\ & + B\{\mu m_1 n_1 + k m_2 n_2 + \mu m_3 n_3 + \nu\,(m_3 n_2 + m_2 n_3) \\ & + \lambda\,(m_3 n_1 + m_1 n_3) + \nu\,(m_1 n_2 + m_2 n_1)\} \\ & + C\{\mu m_1 n_1 + \mu m_2 n_2 + k m_3 n_3 + \nu\,(m_3 n_2 + m_2 n_3) \\ & + \nu\,(m_3 n_1 + m_1 n_3) + \lambda\,(m_1 n_2 + m_2 n_1)\} \\ \equiv {} & U_1 A + U_2 B + U_3 C \text{ suppose} \\ = {} & a\,(U_1 l_1^2 + U_2 l_2^2 + U_3 l_3^2) + b\,(U_1 m_1^2 + U_2 m_2^2 + U_3 m_3^2) \\ & + c\,(U_1 n_1^2 + U_2 n_2^2 + U_3 n^2) + 2f\,(U_1 m_1 n_1 + U_2 m_2 n_2 + U_3 m_3 n_3) \\ & + 2g\,(U_1 n_1 l_1 + U_2 n_2 l_2 + U_3 n_3 l_3) + 2h\,(U_1 l_1 m_1 + U_2 l_2 m_2 + U_3 l_3 m_3),\end{aligned}$$

with similar values for V and W.

In the particular case when the strain is expressed in the form suggested by Thomson and Tait (see VIII. a),

$$l_1 = 1,\ l_2 = 0,\ l_3 = 0;\quad m_1 = 0,\ m_2 = \frac{1}{\sqrt{2}},\ m_3 = \frac{1}{\sqrt{2}},$$

$$n_1 = 0,\ n_2 = \frac{1}{\sqrt{2}},\ n_3 = -\frac{1}{\sqrt{2}}.$$

If a, b, c, and f be the components of the strain, and A, B, C the principal elongations,

$$a = A,\quad b = c = \frac{B+C}{2},\quad f = \frac{B-C}{2},$$

and therefore $\quad A = a,\quad B = \dfrac{b+c+2f}{2},\quad C = \dfrac{b+c-2f}{2},$

therefore $\qquad P_1 = k,\quad P_2 = \mu = P_3,$

therefore $\quad P = kA + \mu\,(B+C) = ka + \mu\,(b+c) = ka + 2\mu b.$

$$Q_1 = \mu + \lambda,\quad Q_1 = \frac{k+\mu+2\nu}{2} = Q_3,$$

therefore $\quad Q = (\mu+\lambda)\,A + \dfrac{k+\mu+2\nu}{2}\,(B+C)$

$$= (\mu+\lambda)\,a + \frac{k+\mu+2\nu}{2}\,(b+c) = (\mu+\lambda)\,a + (k+\mu+2\nu)\,b.$$

$$R_1 = \mu - \lambda,\quad R_2 = \frac{k+\mu-2\nu}{2} = R_3,$$

therefore $R = (\mu-\lambda)A + \frac{k+\mu-2\nu}{2}(B+C)$

$$= (\mu-\lambda)a + \frac{k+\mu-2\nu}{2}(b+c) = (\mu-\lambda)a + (k+\mu-2\nu)b.$$

$$U_1 = 0, \quad U_2 = \frac{k-\mu}{2} = -U_3,$$

therefore $$U = \frac{k-\mu}{2}(B-C) = (k-\mu)f.$$

$$V_1 = 0, \quad V_2 = \frac{\nu-\lambda}{\sqrt{2}} = -V_3,$$

therefore $$V = \frac{\nu-\lambda}{\sqrt{2}}(B-C) = (\nu-\lambda)f\sqrt{2}.$$

$$W_1 = \nu\sqrt{2}, \quad W_2 = \frac{\nu+\lambda}{\sqrt{2}} = W_3,$$

therefore $$W = \nu\sqrt{2}\,A + \frac{\nu+\lambda}{\sqrt{2}}(B+C)$$

$$= \nu\sqrt{2}\,a + (\nu+\lambda)\sqrt{2}\,.\,b.$$

XII. As in the case of the simpler functions of section VII., these transformations may be effected by means of a complex function.

For, if in

$$(kx^2+\mu y^2+\mu z^2+2\lambda yz+2\nu zx+2\nu xy)\,\xi^2$$
$$+(\mu x^2+ky^2+\mu z^2+2\nu yz+2\lambda zx+2\nu xy)\,\eta^2$$
$$+(\mu x^2+\mu y^2+kz^2+2\nu yz+2\nu zx+2\lambda xy)\,\zeta^2$$

the coordinates (x, y, z) be transformed from axes parallel to the principal axes to others, the result will be (if $x = l_1x' + m_1y' + n_1z'$, &c.)

$$(P_1\xi^2+P_2\eta^2+P_3\zeta^2)\,x'^2+(Q_1\xi^2+Q_2\eta^2+Q_3\zeta^2)\,y'^2$$
$$+(R_1\xi_1^2+R_2\eta^2+R_3\zeta^2)\,z'^2+2\,(U_1\xi^2+U_2\eta^2+U_3\zeta^2)\,y'z'$$
$$+2\,(V_1\xi_1^2+V_2\eta^2+V_3\zeta^2)\,z'x'+2\,(W_1\xi^2+W_2\eta^2+W_3\zeta^2)\,x'y'\ ;$$

and if this be again transformed by the equations $\xi = l_1\xi' + m_1\eta' + n_1\zeta'$, &c., the result is

$$\{(P_1l_1^2+P_2l_2^2+P_3l_3^2)\,x'^2+(Q_1l_1^2+Q_2l_2^2+Q_3l_3^2)\,y'^2$$
$$+(R_1l_1^2+R_2l_2^2+R_3l_3^2)\,z'^2+2\,(U_1l_1^2+U_2l_2^2+U_3l_3^2)\,y'z'$$
$$+2\,(V_1l_1^2+V_2l_2^2+V_3l_3^2)\,z'x'+2\,(W_1l_1+W_2l_2^2+W_3l_3^2)\,x'y'\}\,\xi'^2$$
$$+\text{&c.}$$
$$+2\{(P_1m_1n_1+P_2m_2n_2+P_3m_3n_3)\,x'^2+(Q_1m_1n_1+Q_2m_2n_2+Q_3m_3n_3)\,y'^2$$
$$+(R_1m_1n_1+R_2m_2n_2+R_3m_3n_3)\,z'^2+2\,(U_1m_1n_1+U_2m_2n_2+U_3m_3n_3)\,yz'$$
$$+2(V_1m_1n_1+V_2m_2n_2+V_3m_3n_3)\,z'x'$$
$$+2\,(W_1m_1n_1+W_2m_2n_2+W_3m_3n_3)\,x'y'\}\,\eta'\zeta'$$
$$+\text{&c.},$$

where P_1, &c. have the same values as in the last article; so that this is

$$\begin{aligned}&(p_1x'^2+q_1y'^2+r_1z'^2+2u_1y'z'+2v_1z'x'+2w_1x'y')\ \xi'^2\\&\quad+(p_2x'^2+q_2y'^2+\&c.)\ \eta'^2+(p_3x'^2+q_3y'^2+\&c.)\ \zeta'^2\\&\quad+2\ (p_4x'^2+q_4y'^2+r_4z'^2+2u_4y'z'+2v_4z'x'+2w_4x'y')\ \eta'\zeta'\\&\quad+\&c.,\end{aligned}$$

or

$$\begin{aligned}&(p_1\eta'^2+p_2\eta'^2+p_3\zeta'^2+2p_4\eta'\zeta'+2p_5\zeta'\xi'+2p_6\xi'\eta')\ x'^2\\&\quad+(q_1\xi'^2+\&c.)\ y'^2+(r_1\xi'^2+\&c.)\ z'^2\\&\quad+2\ (u_1\xi'^2+u_2\eta'^2+u_3\zeta'^2+2u_4\eta'\zeta'+2u_5\zeta'\xi'+2u_6\xi'\eta')\ y'z'\\&\quad+2\ (v_1\xi'^2+\&c.)\ z'x'+2\ (w_2\xi'^2+\&c.)\ x'y',\end{aligned}$$

where

$$\begin{array}{ccc}P=p_1a+p_2b+p_3c+2p_4f+2p_5g+2p_6h,\\ \&c. \qquad\quad \&c. \qquad\quad \&c.\end{array}$$

This method of transformation applies whatever initial values of the coefficients are adopted. Also, since $Pda+\&c.$ is an exact differential, it follows that the determinant

$$\begin{vmatrix} p_1 \cdot & p_2 \cdot & p_3 \cdot & p_4 \cdot & p_5 \cdot & p_6 \\ q_1 \cdot & q_2 \cdot & q_3 \cdot & q_4 \cdot & q_5 \cdot & q_6 \\ \cdots & \cdots & \cdots & \cdots & & \end{vmatrix}$$

must be symmetrical, as it is with the values already found for the principal coefficients.

[Since the paper, from which the above is expanded, was read before the Society, my attention has been drawn to the fact that Professor C. Niven had, in a paper read to the Royal Society of Edinburgh (*Transactions*, Vol. 27, p. 473), anticipated me in noticing the existence of the invariants D, S, and Δ, and some of the consequences of this.]

In the Recess, Cartes-de-Visite-likenesses of Miss C. A. Scott, and of Profs. Donkin and Rankine (the last two presented by Mr. R. F. Scott), were received.

The following books were also presented to the library:—

"Annali di Matematica," Serie ii^a, Tomo xi., Fasc. 1°, Luglio, 1882.

"American Journal of Mathematics," Vol. iv., No. 4, Dec. 1881; Baltimore, 1882.

"Di alcuni Integrali Formati dagl' Integrali Ellittici e di qualche loro applicazione," Nota di G. Torelli; Napoli, 1873.

"Intorno agl' Integrali Ellittici considerati come funzioni del modulo," per G. Torelli. (No date.)

"Sulle funzioni simmetriche complete e semplice," per G. Torelli. (No date.)

"Sopra alcune proprietà numeriche," Memoria per G. Torelli; Napoli, 1878.

"Sui Determinanti Circolanti," Nota di G. Torelli.

"A Treatise on the Theory of Determinants, with Graduated Lists of Exercises, for use in Colleges and Schools," by Thomas Muir, M.A., F.R.S.E.; London, 1882 (from the Author).

"Archiv for Mathematik og Naturvidenskab" (S. Lie, W. Müller, and G. O. Sars), Sjette Bind, Tredie Hefte, Fjerde Hefte ; Kristiania, 1882.

"Transactions of the Connecticut Academy of Arts and Sciences," Vol. iv., Pt. 2, Vol. v., Pt. 2 ; Newhaven, 1882.

"Annual Report of the Board of Regents of the Smithsonian Institution for the year 1880 ;" Washington, 1881.

"Atti del Reale Istituto Veneto di Scienze, Lettere ed Arti," Tomo ottavo, Serie quinta, Dispensa prima, seconda, terza, quarta, quinta, sesta ; Venezia, 1881-82.

"Bulletin de la Société Mathématique de France," Tome ii., 1, 2, 3 ; v., 2 ; vii., 5 ; viii., 2, 3, 4, 6 ; ix., 1, 2, 4 ; x., 1, 2, 4.

"Transactions of the Royal Irish Academy—Science," Vol. xxviii., vi. (Oct., 1881), vii. (Nov., 1881), viii., ix. (April, 1882), x. (June, 1882).

"Proceedings of the Royal Irish Academy—Science," Vol. iii., Series ii., No. 7 (Dec., 1881), No. 8 (May, 1882). "Polite Literature and Antiquities," Vol. ii., Series ii., No. 3 (Dec., 1881).

Carr's "Synopsis of Results in Pure and Applied Mathematics," Vol. i. and §§ viii. and ix. ; Hodgson, 1882.

"Crelle," 92 Band, 3es, 4es Heft.

"Educational Times," July, 1882.

"Bulletin des Sciences Mathématiques et Astronomiques," Tome vi., Janvier, 1882.

"Atti della R. Accademia dei Lincei—Transunti," Vol. vi., Fasc. 11°, Aprile 1882.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome xvii., 1me, 2me Livr. ; Harlem, 1882.

"The Proceedings of the Royal Society," Vol. xxxiv., No. 220.

"Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin," i.—xvii. ; Berlin, 1882.

"Grundriss einer elementar-geometrischen Kegelschnittslehre," von H. G. Zeuthen ; Leipzig, 1882.

"Bulletin Astronomique et Météorologique de l'Observatoire Impérial de Rio de Janeiro," Janvier (No. 1), Févr. (No. 2), Mars (No. 3) ; Rio, 1882.

"The Physical Society of London—Proceedings," Vol. v., Pt. i., January to June, 1882 ; London, 1882.

"Beiblätter zu den Annalen der Physik und Chemie," Band vi., Stück vi. ; Leipzig, 1882.

"Jahrbuch über die Fortschritte der Mathematik," zwölfter Band, Jahrgang 1880, Heft 1 ; Berlin, 1882.

APPENDIX.

THE joint paper by Messrs. Jenkins and Merrifield (p. 3) was withdrawn, in consequence of its having been pointed out that the authors had been anticipated in their result by M. Lejeune-Dirichlet, in a paper "Ueber die Bestimmung . . . . in der Zahlen Theorie" (see the Berlin *Abhandlungen*, for 1849, pp. 77, &c. ; the *Monatsbericht* for 1838 contains also a paper on the same subject by the same writer).

"Henry Goodwyn, of Blackheath, published in 1818 a table of the decimal values of all different proper fractions whose denominator does not exceed 100, and Sir George Airy* has recently published a table of the logarithms of the reciprocals of the same fractions. The number of such fractions is 3043. Goodwyn has also given an enumeration of the similar series of fractions whose denominator does not exceed 1000. The total number in this case is 304,191. It is easily found by actual enumeration that, when $x = 10$, the number is 31.

"It will be observed that, if we call n the number of such fractions, $n : x^2$ approaches a fixed limit as x increases.

"In fact, we have for

$$x = 10, \quad n/x^2 = 0{\cdot}31; \quad x = 100, \quad n/x^2 = 0{\cdot}3043;$$
$$x = 1000, \quad n/x^2 = 0{\cdot}304191.$$

"It seemed interesting to examine what the limit was when x increased indefinitely. We find the limiting value to be

$$x = \infty, \quad n/x^2 = 3/\pi^2 = 0{\cdot}30396355."$$

The method employed by our authors appears, however, to be quite different from that of Dirichlet.

Mr. Tucker's communication (p. 85) has been published in the *Quarterly Journal of Mathematics* (No. 72, Vol. xviii., pp. 311, 312), and is entitled "The Radial of an Ellipse." The article, *inter alia*, corrects some errors in *Proc.*, Vol. i., No. 5.

Two papers by M. Liguine may be noted here:—(1) "Liste des travaux sur les Ovales de Descartes," *Bulletin des Sciences Mathématiques et Astronomiques*, Tome vi., Févr. 1882, p. 40, &c.; (2) "Sur les Systèmes Articulés de MM. Peaucellier, Hart et Kempe," *Nouvelles Annales*, Avril, 1882, pp. 153—163.

The results of Prof. Wolstenholme's paper (p. 109) are embodied in Question 7053 of the *Educational Times* (May, 1882).

The paper "On Binomial Biordinals" (p. 109) has been withdrawn by the author.

The following corrections have been supplied by the respective authors:—

Vol. xi., p. 126, line 2, the first $\sqrt{-1}$ should be taken negatively; p. 157, line 15, *for* on *read* of.

Vol. xii., p. 65, foot of page, *for* Vol. xiii. *read* Vol. xii.; p. 68, Art. 53, line 8, *for* $-J, U, I$, *read* $-J, U, -I$.

[* Proceedings of the Institution of Civil Engineers, Vol. lxv., Pt. iii., Paper No. 1796. With this paper should be read an article by Sir George Airy, in the *Phil. Mag.* for September, 1881, "On a Systematic Interruption in the Order of Numerical Values of Vulgar Fractions, when arranged in a Series of Consecutive Magnitudes," pp. 175—8.]

Vol. xiii., p. 58, line 4 up, on right-hand side, *read* P/ρ *for* P; p. 59, equation (62), the factor x should be inserted on left-hand side; p. 59, equation (63), *for* ϖ_n *read* $[\varpi_n]$; p. 59, equation (66), the factor ψ_n has been omitted from the first term within the square brackets; p. 60, equation (69), the sign connecting the two terms included within the large bracket on its right-hand side *should be* $-$ *instead of* $+$; pp. 61, 62, in equations (75) and (79), *read* ζ *for* ξ; p. 62, equation (80), the right-hand side should be divided by i; also Vol. xiii., p. 109, line 11 up, *for* confocal *read* cuspidal; and p. 154, line 2, *read* $\gamma^2 = \Omega$.

Prof. W. Stanley Jevons, F.R.S., born in 1835, was drowned at Bexhill, August 13th, 1882. He received the best part of his education at University College, London, having been greatly under the influence of the late Prof. De Morgan, "to whom he owed that capacity for handling mathematical conceptions which distinguishes his best work."*

Being appointed to the Sydney University, he went out to Australia in 1855, thence he returned to England in 1862, and obtained the Gold Medal for Philosophy in the M.A. Examination of the University of London; in 1866 he became Professor of Logic, Political Economy, and Mental Philosophy, at Owens College, Manchester; and in 1875 he was elected to the chair of Political Economy at University College. He subsequently gave up this Professorship to devote himself entirely to his own work.

Dr. William Robertson was born in Edinburgh on the 8th of January, 1818. He was educated at the Edinburgh Academy, and University, and, after completing the medical curriculum, he continued his studies at Paris and Vienna. In 1839 he graduated M.D. of Edinburgh, and four years afterwards was admitted a Fellow of the Royal College of Physicians. He was at one time editor of the *Edinburgh Monthly Journal of Medical Science*, to which he contributed several papers. In 1874, he was appointed to the post of Superintendent of the Statistical Department in the General Registry Office of Births, Deaths, and Marriages, having previously acted as Medical Registrar for Scotland. Dr. Robertson enjoyed a high reputation as a Mathematician and Statist; he was also well versed in classical as well as modern literature, and was an excellent linguist, being familiar with French, German, Italian, and Arabic, and possessing a fair acquaintance with Spanish and Dutch. Dr. Robertson became a Fellow of the Royal Society of Edinburgh in 1860, and a Member of this Society in April, 1867. His death occurred at his residence in Albany Street, Edinburgh, on the 25th of August, 1882.†

R. T.

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* *Athenæum*, No. 2860, August 19th, 1882.

† The above facts respecting Dr. Robertson are taken from an obituary notice put at our disposal by Prof. Tait, of Edinburgh.

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